The Discrete Quasi-Eigenfunction Approximation

Joseph P. Havlicek †, *Alan C. Bovik* †, *Mita D. Desai* ‡, *and David S. Harding* †

†Laboratory for Vision Systems, University of Texas, Austin, TX 78712-1084
‡Division of Engineering, University of Texas, San Antonio, TX 78729

Abstract

We develop powerful approximations for the response of discrete-time linear systems with wideband, nonstationary inputs modeled as AM-FM functions. On a temporally local basis, the approximations are analogous in form to the eigenfunction interpretation of sinusoids in linear system theory, and hence are called *quasi-eigenfunction approximations.* Using these approximations, we obtain straightforward solutions to several difficult problems in discrete AM-FM signal demodulation. We bound the approximation error by functionals quantifying the time duration of the system unit pulse response and the local coherency of the input, and for the first time study the variation of the bound with the amplitude and frequency modulation signals.

1 Introduction

The analysis of wideband, nonstationary signals in terms of AM-FM modulation models is rapidly becoming a standard technique [1–7]. AM-FM modeling is most useful when the signals of interest may be accurately represented as a sum of one or more locally coherent complex-valued components, each of the form

$$\xi(k) = a(k)\exp\left[j\varphi(k)\right], \tag{1}$$

where $\xi : \mathbb{Z} \to \mathbb{C}$ and $a, \varphi : \mathbb{Z} \to \mathbb{R}$, or a sum of locally coherent real-valued components

$$x(k) = a(k)\cos\left[\varphi(k)\right]. \tag{2}$$

In (1),(2), we assume that $\xi(k)$, $x(k)$, $a(k)$, and $\varphi(k)$ are samples of continuous domain functions $\xi(t)$, $x(t)$, $a(t)$, and $\varphi(t)$, taken with unity sampling interval. Note that $\xi(t)$ and $x(t)$ are uniquely related through $x(t) = \mathrm{Re}[\xi(t)]$ and $\xi(t) = x(t) + j\boldsymbol{\mathcal{H}}\left[x(t)\right]$, where $\boldsymbol{\mathcal{H}}[.]$ indicates Hilbert transform. By *locally coherent*, we mean that the amplitude and frequency modulating signals $a(t)$ and $\varphi'(t)$ do not vary too wildly or rapidly with relation to the sampling interval (this notion will be made explicit in Section 2.1). Despite being globally wideband, such signals may be considered narrowband on a local scale. Fractal, self-similar, or grotesquely discontinuous signals, which are termed *incoherent* in this sense, are best treated by other methods.

Given $\xi(k)$, or equivalently $x(k)$, the discrete AM-FM demodulation problem is concerned with estimating the amplitude modulation $a(k)$ and the *emergent*, or *instantaneous*, frequencies $\varphi'(k)$ that characterize the structure of the signal on a temporo-spectrally local basis, where $\varphi'(k) = \frac{d}{dt}\varphi(t)\left.\right|_{t=k}$. When multiple components are present, it becomes necessary to isolate the individual components from one another by linear filtering prior to demodulation. Without introducing certain simplifying approximations, the rigorous development and analysis of discrete demodulation algorithms for components of the form (1),(2) is difficult, and it is generally impossible for filtered components. In this paper we present a powerful approximation to the response of general discrete linear systems to signals modeled with AM-FM functions, and demonstrate how the approximation can be used with great efficacy in treating discrete demodulation schemes. On

This research was supported in part by a grant from the Texas Advanced Research Projects Agency and by the Air Force Office of Scientific Research, Air Force Systems Command, USAF, under grant number F49620-93-1-0307.

a temporally local scale, the approximation bears similarities to the eigenfunction interpretation of the complex exponential in linear system theory, and hence has come to be known as the *quasi-eigenfunction approximation* (hereafter QEA). For the first time, we also study behavior of the approximation error bound.

2 Approximate Linear System Response

In this section, we approximate the response of square-summable discrete linear systems to AM-FM inputs, bound the approximation error, and show examples applying the complex-valued approximation. Consider an arbitrary discrete linear system $g : \mathbb{Z} \to \mathbb{C}$ with unit pulse response $g(k) \in \ell^2(\mathbb{Z})$ and frequency response

$$G\left(e^{j\omega}\right) = \mathcal{F}\left[g(k)\right] = \sum_{n \in \mathbb{Z}} g(n) e^{-j\omega n}. \tag{3}$$

Henceforth, we write $g(k) \Leftrightarrow G\left(e^{j\omega}\right)$ to mean that $g(k)$ and $G\left(e^{j\omega}\right)$ are a Fourier transform pair. The response of g to input $\xi(k)$ is given exactly by

$$\zeta(k) = g(k) * \xi(k) = \sum_{n \in \mathbb{Z}} g(n)\xi(k-n). \tag{4}$$

Without specific knowledge of the filter function, simplifications beyond (4) are generally impossible. However, if the modulating functions are sufficiently slowly varying (*i.e.* if $\xi(k)$ is sufficiently locally coherent), and if $g(k)$ is sufficiently temporally localized, then on a *local* basis we expect that $\xi(k)$ does closely resemble an eigenfunction. This motivates the QEA

$$\widehat{\zeta}(k) = \xi(k) G\left(e^{j\varphi'(k)}\right) = a(k) \exp\left[j\varphi(k)\right] G\left(e^{j\varphi'(k)}\right). \tag{5}$$

For an arbitrary real-valued discrete linear system $h : \mathbb{Z} \to \mathbb{R}$ with unit pulse response $h(k) \in \ell^2(\mathbb{Z})$ and frequency response $H\left(e^{j\omega}\right) \Leftrightarrow h(k)$, the exact response to input $x(k)$ is given by

$$y(k) = h(k) * x(k) = \sum_{n \in \mathbb{Z}} h(n) x(k-n), \tag{6}$$

while the QEA to the response is

$$\widehat{y}(k) = a(k) \left| H\left(e^{j\varphi'(k)}\right)\right| \cos\left\{\varphi(k) + \angle H\left(e^{j\varphi'(k)}\right)\right\}. \tag{7}$$

Although it is true that an arbitrary AM-FM signal may bear little resemblance to a linear system eigenfunction on a *global* scale, it is important to observe that all of the quantities involved in (5) and (7) are in fact localized. Except for the special case of $a(k)$ and $\varphi'(k)$ constant, there will always be an error in the QEA. However, when the duration of $g(k)$ is short with relation to the nonstationary features of $\xi(k)$, the approximation error is generally insignificant. We bound the error in Section 2.1.

2.1 Approximation Error Bounds

Various instances of QEA error bounds have appeared previously. The 1-D continuous domain version was proven in [8], the 2-D version in [9], the n-D version for the case of *FM-only* signals in [10], and the general n-D case in [11]. The 1-D discrete domain bound was treated in [5]. Proof of the discrete domain n-D theorem remains an important open problem. Here, in stating the bound, we will make use of several definitions. First, denote the maximum value of $a(k)$ by $a_{\max} = \sup_{k \in \mathbb{Z}} |a(k)|$. Quantify the time duration of the linear system by its k^{th} energy moment functional

$$\Delta_k(g) = \left[\sum_{n \in \mathbb{Z}} n^{2k} \left|g(n)\right|^2\right]^{\frac{1}{2}} = \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\frac{d^k}{d\omega^k} G\left(e^{j\omega}\right)\right|^2 d\omega\right]^{\frac{1}{2}}. \tag{8}$$

Note that $\Delta_k(g)$ tends to increase with the duration of $g(k)$. While it is true that the duration of $G\left(e^{j\omega}\right)$ grows inversely with that of $g(k)$, observe that the frequency domain relationship in (8) is through a Sobolev norm. The degree to which the input signal is locally coherent is quantified by functionals of the form

$$\mathcal{D}(a;k) = |a'(k)| + \sqrt{\gamma}\int_{\mathbb{R}} |a'(u)|du, \tag{9}$$

where the integral is again a Sobolev norm, and

$$\sqrt{\gamma} = \left[2\sum_{n=1}^{\infty}\frac{1}{n^2}\right]^{\frac{1}{2}} = \frac{\pi}{\sqrt{6}} \approx 1.2825 \tag{10}$$

is the Riemann Zeta function. The following theorem bounds the error in the QEA (11):

Theorem Suppose a is differentiable, φ is twice differentiable, $a'(t)$, $\varphi''(t) \in L^1(\mathbb{R})$, and $a(k)$, $a'(k)$, $\varphi''(k) \in \ell^\infty(\mathbb{Z})$. Let $\varepsilon_\zeta(k) = |\zeta(k) - \widehat{\zeta}(k)|$, where $\zeta(k)$ and $\widehat{\zeta}(k)$ are given by (4) and (5), and let $kg(k)$, $k^2 g(k) \in \ell^2(\mathbb{Z})$. Then

$$\varepsilon_\zeta(k) \le a_{\max}\Delta_2(g)\,\mathcal{D}(\varphi';k) + \Delta_1(g)\,\mathcal{D}(a;k)\,. \tag{11}$$

Next, we give an analogous bound for the error in (7):

Corollary Take a and φ as in the preceding Theorem. Let $\varepsilon_y(k) = |y(k) - \widehat{y}(k)|$, where $y(k)$ and $\widehat{y}(k)$ are given by (6) and (7), and let $kh(k), k^2h(k) \in \ell^2(\mathbb{Z})$. Then

$$\varepsilon_y(k) \le a_{\max}\Delta_2(h)\,\mathcal{D}(\varphi';k) + \Delta_1(h)\,\mathcal{D}(a;k)\,. \tag{12}$$

In (11),(12), the error is bounded by terms involving the duration of the system unit pulse response and the smoothness of the modulating functions, which is in agreement with the preceding intuitive discussions. As the input tends toward a true eigenfunction, the bounds become tight in the sense that $\mathcal{D}(a;k)$ and $\mathcal{D}(\varphi';k)$ tend to zero.

2.2 Examples

In this section we present examples demonstrating behavior of the QEA error bound (11) applied to a Gabor filter with center frequency f_g Hz/sample and time constant σ_g,

$$g(k) = \left(2\pi\sigma_g^2\right)^{-\frac{1}{4}} \exp\left(\frac{-k^2}{4\sigma_g^2}\right) \exp\left(j2\pi f_g k\right), \tag{13}$$

when the input amplitude modulation is Gaussian and the instantaneous phase $\varphi(k)$ is the product of a quadratic (linear chirp characteristic) and a Gaussian. The form of the input is

$$\xi(k) = \exp\left(\frac{-k^2}{2\sigma_a^2}\right) \exp\left[j\left(\frac{\pi\Delta_f}{S}k^2 + 2\pi f_i k\right)\exp\left(\frac{-k^2}{2\sigma_f^2}\right)\right], \tag{14}$$

where σ_a is the AM time constant, σ_f is the time constant of the Gaussian phase component, the FM chirp component progresses from an initial frequency of f_i Hz/sample to a final frequency of f_f Hz/sample in S seconds, and $\Delta_f = f_f - f_i$ is the frequency differential. Note that $\xi(k)$ tends toward a true eigenfunction in the limit as $\sigma_a, \sigma_f \to \infty$ and $\Delta_f \to 0$.

For a filter specified by $f_g = 0.4$ and $\sigma_g = 0.26$, Figure 1(a) shows the variation of the QEA error bound with σ_a for $\sigma_f = 2048$, $S = 1024$, and $f_f = f_i = 0.4$. Variation of the bound with Δ_f for $\sigma_a = 1024$, $\sigma_f = 2048$, $S = 1024$, and $f_i = 0.4 \ge f_f$ is shown in Figure 1(b). In all cases, the bound is dominated by the Sobolev norms in $\mathcal{D}(a;k)$ and $\mathcal{D}(\varphi';k)$, which measure the smoothness of the infinitely supported modulating functions on a global scale. Indeed, the local variations of each bound are small in relation to their mean values. The real parts of the exact and approximated responses for $f_f = 0.1$ are shown in the (c) and (d) parts of the figure, respectively, while the bound and the actual error $\varepsilon_\zeta(k)$ are shown in the (e) and (f) parts. In this case, the peak error is four orders of magnitude below the amplitude of the response. We have found that for reasonably localized filters and reasonably locally coherent inputs, the approximation is typically excellent. In fact, the error shown in Figure 1(f) is larger than that which occurred in any of the examples shown in the (a) and (b) parts of the figure except for the case $\Delta_f = -0.4$, where the peak error was 7.5×10^{-5}.

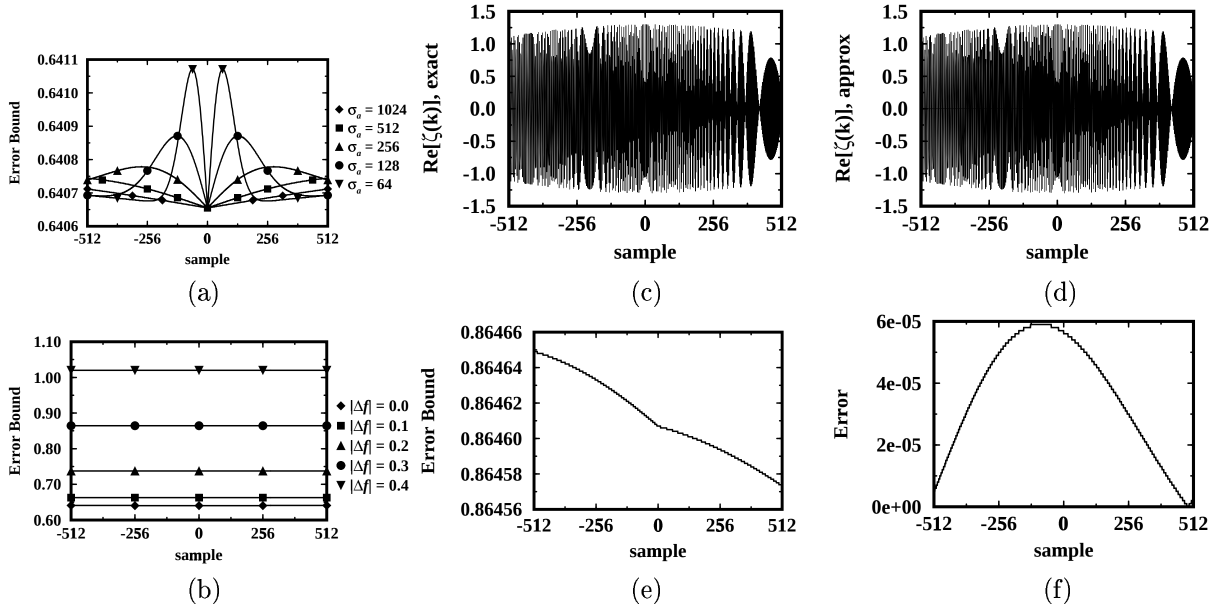

Figure 1: Behavior of the QEA error bound. (a) variation of the bound with AM for fixed FM, (b) variation of the bound with FM for fixed AM, (c) exact response for $\Delta_f = -0.3$, (d) approximated response, (e) error bound, (f) $\varepsilon_\zeta(k)$, magnitude of the actual error.

3 Application to the Discrete AM-FM Demodulation Problem

In this section we use the discrete QEA (5) to develop two demodulation algorithms for complex-valued AM-FM signals of the form (1). Consider a discrete domain linear system parameterized by constants $n_1, n_2 \in \mathbb{Z}$ and $q = \pm 1$, where the unit pulse response and frequency response of the system are

$$g(k) = \delta(k+n_1) + q\delta(k+n_2) \Leftrightarrow G\left(e^{j\omega}\right) = e^{j\omega n_1} + qe^{j\omega n_2}. \tag{15}$$

The exact response is

$$\zeta(k) = g(k) * \xi(k) = \xi(k+n_1) + q\xi(k+n_2), \tag{16}$$

while the QEA is

$$\widehat{\zeta}(k) = \xi(k)\left[e^{jn_1\varphi'(k)} + qe^{jn_2\varphi'(k)}\right]. \tag{17}$$

Upon equating (16) and (17) subject to the approximation error, with $n_1 = 1$ and $n_2 = q = -1$ we have almost immediately that

$$\varphi'(k) \approx \widehat{\varphi}'(k) = \arcsin\left[\frac{\xi(k+1) - \xi(k-1)}{2j\xi(k)}\right]. \tag{18}$$

Alternatively, choosing $n_1 = q = 1$ and $n_2 = -1$, we have

$$\varphi'(k) \approx \widehat{\varphi}'(k) = \arccos\left[\frac{\xi(k+1) + \xi(k-1)}{2\xi(k)}\right]. \tag{19}$$

Using the QEA it was straightforward to obtain these frequency demodulation algorithms from (15), whereas a naive discretization of the analogous continuous domain algorithms would fail to predict the presence of the transcendentals. Since arcsin and arccos are not single-valued, there is some ambiguity in estimating $\varphi'(k)$ from either (18) or (19) alone. However, they can be used together to correctly place $\widehat{\varphi}'(k)$ within 2π radians. Once $\widehat{\varphi}'(k)$ is known, $a(k)$ is estimated by $\widehat{a}(k) = |\xi(k)|$.

3.1 Multi-Component Demodulation

In many cases where no representation of the form (1) admits smooth modulating functions, the signals under analysis may be best modeled as a sum of locally coherent components. Due to the highly nonlinear nature of the algorithms (18),(19), it then becomes necessary to isolate the multiple components by linear filtering prior to demodulation. In this section, we use the QEA to develop a demodulation scheme for the resulting *filtered* AM-FM signal components. Let $f(k) \Leftrightarrow F\left(e^{j\omega}\right)$ be a discrete linear filter, and consider the response of the cascade system $f(k) * g(k)$ to input $\xi(k)$, where $g(k)$ is given by (15). The frequency response of the cascade system is $F\left(e^{j\omega}\right)\left[e^{j\omega n_1} + qe^{j\omega n_2}\right]$. Define $\lambda(k) = \xi(k) * f(k)$, and $\zeta(k) = \lambda(k) * g(k) = \xi(k) * f(k) * g(k)$. The exact response of G in terms of $\lambda(k)$ can be obtained directly from the convolution sum:

$$\zeta(k) = \sum_{n \in \mathbb{Z}} \lambda(n) g(k-n) = \lambda(k+n_1) + q\lambda(k+n_2). \tag{20}$$

Apply the QEA to the cascade system to obtain

$$\widehat{\zeta}(k) = \xi(k) F\left(e^{j\varphi'(k)}\right)\left[e^{jn_1\varphi'(k)} + qe^{jn_2\varphi'(k)}\right] \approx \lambda(k)\left[e^{jn_1\varphi'(k)} + qe^{jn_2\varphi'(k)}\right], \tag{21}$$

where the approximate equality is obtained by applying the QEA to F alone. Upon equating (20) and (21) to within the approximation error, we obtain

$$\lambda(k+n_1) + q\lambda(k+n_2) \approx \lambda(k)\left[e^{jn_1\varphi'(k)} + qe^{jn_2\varphi'(k)}\right]. \tag{22}$$

Taking $n_1 = 1$, $n_2 = q = -1$, (22) immediately validates applying (18) directly to a filtered component:

$$\widehat{\varphi'}(k) = \arcsin\left[\frac{\lambda(k+1) - \lambda(k-1)}{2j\lambda(k)}\right]. \tag{23}$$

Likewise, choosing $n_1 = q = 1$, $n_2 = -1$, establishes that (19) can be applied directly to $\lambda(k)$:

$$\widehat{\varphi'}(k) = \arccos\left[\frac{\lambda(k+1) + \lambda(k-1)}{2\lambda(k)}\right]. \tag{24}$$

Once $\widehat{\varphi'}(k)$ has been obtained, the amplitude is estimated by

$$\widehat{a}(k) = \left|\frac{\lambda(k)}{G\left(e^{j\widehat{\varphi'}(k)}\right)}\right|. \tag{25}$$

Figure 2 shows demodulation of a filtered chirp signal with Gaussian AM using (23) and (24) together. The frequency of the chirp varies from 0.4 to 0.25 Hz/sample. $\mathrm{Re}[\xi(k)]$ is shown in the (a) part of the figure. The filter is a half-octave Gabor wavelet with center frequency $f_g = 0.375$, the real part of which is shown in Figure 2(b). The estimated frequency and amplitude modulating functions are shown in the (c) and (d) parts of the figure, while the estimation errors are shown in the (e) and (f) parts. The peak errors are on the order of one percent of the signal amplitude.

4 Conclusions and Future Work

Used properly, quasi-eigenfunction approximations have powerful and natural application for the analysis of discrete linear systems with AM-FM inputs, and have been employed to effect a rigorous analysis of the Teager-Kaiser energy operator [5]. We demonstrated that several demodulation problems which are difficult or impossible without the approximation become straightforward when the QEA is applied, and also gave practical examples where the approximation error was negligible. We bounded the error by functionals quantifying the duration of the filter and the local coherency of the input signal, but observed that, in practice, the actual errors tended to be well below the bound, which is tight only in the sense that it vanishes as the input tends toward a true eigenfunction. The bound (11) was dominated by Sobolev norms of the continuous domain modulating functions, which are global measures of smoothness and reflect the fact that $a(t)$ and $\varphi'(t)$ are infinitely supported. In practical applications, the support of the modulating functions is always finite. Furthermore, in the computation of a given response sample, the *effective support* of the input affecting the response can be no larger than that of the filter unit pulse response, and future work will use these facts to achieve a tighter error bound.

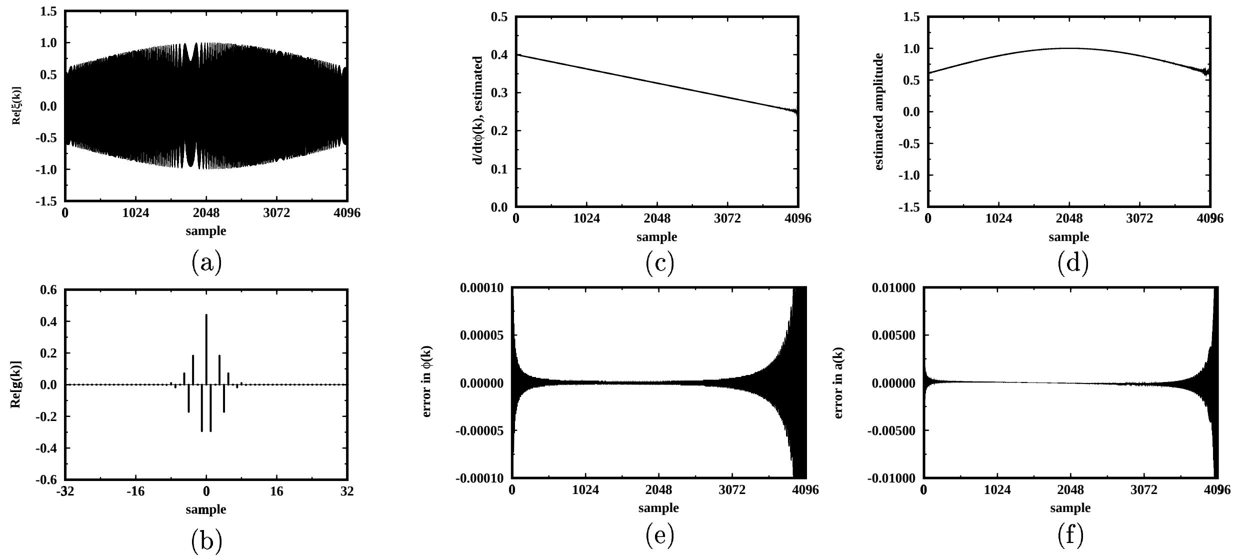

Figure 2: Filtered demodulation example. (a) chirp with Gaussian AM, (b) real part of the filter, (c) $\widehat{\varphi}'(k)$, (d) $\widehat{a}(k)$, (e) frequency estimation error, (f) amplitude estimation error.

References

[1] H. M. Teager and S. M. Teager, "Evidence for nonlinear sound production mechanisms in the vocal tract", in *Speech Production and Speech Modelling*, W. J. Hardcastle and A. Marchal, Eds., vol. 55 of *NATO Advanced Study Institute Series D*, pp. 241–261. Kluwer Academic Publishers, Boston, MA, 1990.

[2] A. Reilly, G. Frazer, and B. Boashash, "Analytic signal generation – tips and traps", *IEEE. Trans. Signal Proc.*, vol. SP-42, no. 11, pp. 3241–3245, November 1994.

[3] P. Maragos, J. F. Kaiser, and T. F. Quatieri, "On amplitude and frequency demodulation using energy operators", *IEEE Trans. Signal Proc.*, vol. SP-41, no. 4, pp. 1532–1550, April 1993.

[4] P. Maragos, J. F. Kaiser, and T. F. Quatieri, "Energy separation in signal modulations with applications to speech analysis", *IEEE. Trans. Signal Proc.*, vol. SP-41, no. 10, pp. 3024–3051, October 1993.

[5] A. C. Bovik, J. P. Havlicek, and M. D. Desai, "Theorems for discrete filtered modulated signals", in *Proc. IEEE Int'l. Conf. Acoust., Speech, Signal Proc.*, Minneapolis, MN, April 27 - 30 1993, pp. III153–III156.

[6] J. P. Havlicek, D. S. Harding, and A. C. Bovik, "Multi-component signal demodulation and reconstruction using AM-FM modulation models", in *Proc. 1995 IEEE Workshop Nonlin. Signal and Image Proc.*, Neos Marmaras, Halkidiki, Greece, June 20 - 22 1995.

[7] L. Cohen, *Time-Frequency Analysis*, Prentice Hall, Englewood Cliffs, NJ, 1995.

[8] A. C. Bovik, P. Maragos, and T. F. Quatieri, "AM-FM energy detection and separation in noise using multiband energy operators", *IEEE. Trans. Signal Proc.*, vol. SP-41, no. 12, pp. 3245–3265, December 1993.

[9] A. C. Bovik, N. Gopal, T. Emmoth, and A. Restrepo, "Localized measurement of emergent image frequencies by Gabor wavelets", *IEEE. Trans. Info. Theory*, vol. IT-38, no. 2, pp. 691–712, March 1992.

[10] A. C. Bovik, "A bound involving n-dimensional instantaneous frequency", *IEEE. Trans. Circ. Syst.*, vol. CAS-38, no. 11, pp. 1389–1390, November 1991.

[11] J. P. Havlicek and A. C. Bovik, "AM-FM models, the analytic image, and nonlinear demodulation techniques", Tech. Rept. TR-95-001, Center for Vision and Image Sciences, The University of Texas at Austin, March 1995.