The Analytic Image

Joseph P. Havlicek †, John W. Havlicek ††, and Alan C. Bovik ‡

†School of Electrical and Computer Engineering, The University of Oklahoma, Norman, OK 73019-1023
††Department of Computer Sciences, The University of Texas, Austin, TX 78712
‡Center for Vision and Image Sciences, The University of Texas, Austin, TX 78712-1084

Abstract

We introduce a novel directional multidimensional Hilbert transform and use it to define the complex-valued analytic image associated with a real-valued image. The analytic image associates a unique pair of instantaneous amplitude and frequency functions with an image, and also admits many of the other important properties of the one-dimensional analytic signal.

1. Introduction

The one-dimensional *analytic signal* has been used in communications engineering, physics, and signal analysis since it was introduced by Gabor in 1946 [1]. However, relatively little research has been devoted to extending the notion of analytic signal to multiple dimensions [2–5]. In this paper, we use a novel directional multidimensional Hilbert transform to define a multidimensional analytic signal which we call the *analytic image*. Unlike its one-dimensional counterpart, the analytic image does *not* generally satisfy the multidimensional Cauchy-Riemann equations. However, the analytic image does admit many of the other attractive properties of the one-dimensional analytic signal. Consequently, it may be used with great efficacy in the study of nonstationary, jointly amplitude-frequency modulated sigals, or *AM-FM signals,* which have recently been intensively studied in both one and multiple dimensions [5–8].

Consider a real-valued signal $s(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$, where $\mathbf{x} = [x_1\ x_2\ \ldots\ x_n]^T$. A problem of fundamental importance in engineering and in the pure and applied sciences is that of associating with $s(\mathbf{x})$ an *instantaneous amplitude* function $a(\mathbf{x}) : \mathbb{R}^n \to [0, \infty)$ and an *instantaneous phase* function $\varphi(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ such that $s(\mathbf{x}) = a(\mathbf{x}) \cos[\varphi(\mathbf{x})]$. The *instantaneous frequency* of $s(\mathbf{x})$ is then defined by the vector-valued quantity $\nabla\varphi(\mathbf{x})$. However, corresponding to *any* real-valued signal $s(\mathbf{x})$ there are *uncountably* infinitely many pairs of functions $a(\mathbf{x}),\varphi(\mathbf{x})$ for which $s(\mathbf{x}) = a(\mathbf{x}) \cos[\varphi(\mathbf{x})]$.

In contrast, for any complex-valued signal $z(\mathbf{x}) : \mathbb{R}^n \to \mathbb{C}$ we may consider that $z(\mathbf{x}) = a(\mathbf{x}) \exp[j\varphi(\mathbf{x})]$, where the amplitude $a(\mathbf{x})$ and frequency $\nabla\varphi(\mathbf{x})$ are *unique*. Thus, if $z(\mathbf{x}) = s(\mathbf{x}) + jq(\mathbf{x})$, then *each* choice of $q(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ associates a *unique* amplitude and frequency with $s(\mathbf{x})$.

This research was supported in part by the Army Research Office under contract DAAH 049510494 and by the Air Force Office of Scientific Research, Air Force Systems Command, USAF, under grant number F49620-93-1-0307.

2. 1D Analytic Signal

Gabor [1] and Ville [9] advocated defining the amplitude and frequency of a one-dimensional real-valued signal $s(x) : \mathbb{R} \to \mathbb{R}$ in terms of the complex-valued *analytic signal* defined by $z(x) = s(x) + jq(x)$, where

$$q(x) = \mathcal{H}[s(x)] = s(x) * \frac{1}{\pi x} = \frac{1}{\pi} \int_{\mathbb{R}} \frac{s(\tau)}{x - \tau} d\tau \qquad (1)$$

is the Hilbert transform of $s(x)$.

We now summarize several of the more important properties of the analytic signal $z(x)$. Clearly, $\mathrm{Re}\,[z(x)] = s(x)$. The analytic spectrum $Z(\Omega) = \mathcal{F}[z(x)]$ is supported only on the non-negative half-line, and $Z(\Omega) = 2S(\Omega)$ on the positive half-line. Thus, the spectrum $Z(\Omega)$ corresponds to a full reduction of the spectral redundancy inherent in the conjugate symmetric spectrum $S(\Omega)$. If $w \in \mathbb{C}$ is a complex variable, then $z(w)$ is holomorphic in the upper half-plane.

Suppose that $z(x) = a(x)e^{j\varphi(x)}$ is the analytic signal associated with a real signal $s(x) \in L^2(\mathbb{R})$, and let $\mathcal{E}_0 = ||z(x)||^2_{L^2} = ||Z(\Omega)||^2_{L^2}/2\pi$. Then, the first moment of frequency in $z(x)$ is

$$\overline{\Omega} = \frac{1}{2\pi\mathcal{E}_0} \int_{\mathbb{R}} \Omega |Z(\Omega)|^2 d\Omega. \qquad (2)$$

One may then show that the first moment of instantaneous frequency in $z(x)$ is given by [9–11]

$$\overline{\dot\varphi(x)} = \frac{1}{\mathcal{E}_0} \int_{\mathbb{R}} \dot\varphi(x) |z(x)|^2 dx = \overline{\Omega}; \qquad (3)$$

that is, the first moments of the instantaneous and Fourier frequencies in $z(x)$ are equal. Although the centroid of $|S(\Omega)|$ is zero, our intuition insists that the first moment of frequency in $s(x)$ *should* be precisely the value given by (2), which is the centroid of $|S(\Omega)|$ computed over non-negative frequencies *only.*

Now suppose that $s(x) = \cos(\omega_0 x)$. In this case our intuition strongly suggests that the frequency moments *should* be $\overline{\dot\varphi(x)} = \overline{\Omega} = \omega_0$, and indeed this is exactly the result obtained by applying (2) and (3) to $z(x)$.

0-8186-8183-7/97 $10.00 © 1997 IEEE

3. Analytic Image

Let $s(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ and let $\mathbf{e}_i$ denote a unit vector in the x_i direction. Let $\mathcal{Z} = \{\mathbf{\Omega} : \Omega_1 = 0\}$, where $\mathbf{\Omega} = [\Omega_1\ \Omega_2\ \ldots\ \Omega_n]^T$. The n-dimensional directional Hilbert transform is usually defined by [12]

$$\mathcal{H}_{ord}[s(\mathbf{x})] = \frac{1}{\pi}\int_{\mathbb{R}} s(\mathbf{x}-\xi\mathbf{e}_1)\frac{d\xi}{\xi}. \tag{4}$$

We refer to the transform (4) as the *ordinary* Hilbert transform. It is a multiplier transform, and

$$\mathcal{F}\{\mathcal{H}_{ord}[s(\mathbf{x})]\} = -j\,\mathrm{sgn}(\Omega_1)S(\mathbf{\Omega}). \tag{5}$$

Now, $S(\mathbf{\Omega})$ contains inherent spectral redundancy manifest in pairs of frequency orthants symmetric about the origin. If $s(\mathbf{x})$ belongs to $L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$, then the spectrum of the complex-valued image $s(\mathbf{x}) + j\mathcal{H}_{ord}[s(\mathbf{x})]$ is supported only on $n/2$ orthants, where it is twice $S(\mathbf{\Omega})$. Unfortunately however, $\mathcal{H}_{ord}$ does not lead to a theoretically consistent definition of the analytic image for signals whose spectra admit unit masses on $\mathcal{Z}$. For example, $\mathcal{H}_{ord}\left[\cos(\mathbf{\Omega}^T\mathbf{x})\right] \neq \sin(\mathbf{\Omega}^T\mathbf{x})$ when $\mathbf{\Omega}\in\mathcal{Z}$.

To overcome this problem, we have developed the *new* transform given in the following definition.

Definition The *adjusted* multidimensional Hilbert transform $\mathcal{H}_{adj}$ of a signal $s(\mathbf{x})$ is given by

$$\mathcal{H}_{adj}[s(\mathbf{x})] = \mathcal{F}^{-1}\Big\{M_{adj}(\mathbf{\Omega})S(\mathbf{\Omega})\Big\}. \tag{6}$$

The spectral multiplier $M_{adj}(\mathbf{\Omega})$ is defined by

$$M_{adj}(\mathbf{\Omega}) = -j\,\mathrm{sgn}_{adj}\,\mathbf{\Omega}, \tag{7}$$

where

$$\mathrm{sgn}_{adj}\,\mathbf{\Omega} = \sum_{i=1}^{n}\mathrm{sgn}_*\,\Omega_i\prod_{k=1}^{i-1}\Big(1-\big|\mathrm{sgn}_*\,\Omega_k\big|\Big) \tag{8}$$

and where

$$\mathrm{sgn}_*\,x = \begin{cases} 1, & x>0 \\ 0, & x=0 \\ -1, & x<0. \end{cases} \tag{9}$$

■

Note that $M_{adj}(\mathbf{\Omega})$ differs from the multiplier of $\mathcal{H}_{ord}$ only on $\mathcal{Z}$, which is a set of Lebesgue measure zero. The following theorem articulates the relationship between the transforms $\mathcal{H}_{adj}$ and $\mathcal{H}_{ord}$. The proof is omitted for brevity.

Theorem Let n be finite. Then the following hold.

(*a*) $\mathcal{H}_{adj}$ is a bounded linear operator on $L^2(\mathbb{R}^n)$. Furthermore, for all $s(\mathbf{x})\in L^2(\mathbb{R}^n)$, $\mathcal{H}_{adj}[s(\mathbf{x})] = \mathcal{H}_{ord}[s(\mathbf{x})]$.

(*b*) If $s(\mathbf{x})$ admits a Fourier transform representation $S(\mathbf{\Omega})$ such that $\int_{\mathcal{Z}}|S(\mathbf{\Omega})|\,d\mathbf{\Omega} = 0$ and if $\mathcal{H}_{ord}[s(\mathbf{x})]$ exists, then $\mathcal{H}_{adj}[s(\mathbf{x})] = \mathcal{H}_{ord}[s(\mathbf{x})]$.

(*c*) If $s(\mathbf{x})$ admits a Fourier series representation

$$s(\mathbf{x}) = \sum_{i\in\mathbb{N}} A_i\cos(\mathbf{\Omega}_i^T\mathbf{x}) + B_i\sin(\mathbf{\Omega}_i^T\mathbf{x}), \tag{10}$$

where $A_i, B_i\in\mathbb{R}$ and where $\mathrm{sgn}_{adj}\,\mathbf{\Omega}_i > 0$ for each $i\in\mathbb{N}$, then

$$\mathcal{H}_{adj}[s(\mathbf{x})] = \sum_{i\in\mathbb{N}} A_i\sin(\mathbf{\Omega}_i^T\mathbf{x}) - B_i\cos(\mathbf{\Omega}_i^T\mathbf{x}). \tag{11}$$

■

The theorem indicates that $\mathcal{H}_{adj}$ agrees with $\mathcal{H}_{ord}$ when the signal spectrum does not admit unit masses on $\mathcal{Z}$, and also that $\mathcal{H}_{adj}$ maps *all* sinusoids in the intuitively expected way. If the signal spectrum does admit unit masses on $\mathcal{Z}$, $\mathcal{H}_{ord}$ generally fails to exist. It is important to realize that the Hilbert transform of such a signal does not generally satisfy the sufficiency conditions of the Fubini theorem, and that the Fourier transform convolution theorem therefore is not generally applicable to such Hilbert transforms. Thus, it is meaningless to speak of defining the transform $\mathcal{H}_{adj}$ through a spatial convolution.

For any signal $s(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$, we define the *analytic image* associated with $s(\mathbf{x})$ by $z(\mathbf{x}) = s(\mathbf{x}) + j\mathcal{H}_{adj}[s(\mathbf{x})]$, provided that $\mathcal{H}_{adj}[s(\mathbf{x})]$ exists. With this definition, $\mathrm{Re}[z(\mathbf{x})] = s(\mathbf{x})$ and the spectrum $Z(\mathbf{\Omega})$ is supported on $n/2$ orthants, where it is twice $S(\mathbf{\Omega})$. The analytic image $z(\mathbf{x}) = a(\mathbf{x})\exp[j\varphi(\mathbf{x})]$ associates with $s(\mathbf{x})$ a unique instantaneous amplitude $a(\mathbf{x})$ and instantaneous frequency $\nabla\varphi(\mathbf{x})$. If $s(\mathbf{x})\in L^2(\mathbb{R}^n)$ and $\mathcal{E}_0 = ||z(\mathbf{x})||^2_{L^2}$, then

$$\overline{\nabla\varphi(\mathbf{x})} = \frac{1}{\mathcal{E}_0}\int_{\mathbb{R}^n}\nabla\varphi(\mathbf{x})|z(\mathbf{x})|^2 d\mathbf{x} \tag{12}$$

$$= \frac{1}{(2\pi)^n\mathcal{E}_0}\int_{\mathbb{R}^n}\mathbf{\Omega}|Z(\mathbf{\Omega})|^2 d\mathbf{\Omega} = \overline{\mathbf{\Omega}}, \tag{13}$$

where $\overline{\mathbf{\Omega}}$ is the centroid of $|S(\mathbf{\Omega})|$ computed over $n/2$ nonredundant orthants; this is precisely what our intuition insists that the mean frequency of $s(\mathbf{x})$ *should* be. Furthermore, for every signal $s(\mathbf{x}) = \cos(\mathbf{\Omega}_0^T\mathbf{x})$, we have the intuitively satisfying result that $\nabla\varphi(\mathbf{x}) = \overline{\nabla\varphi(\mathbf{x})} = \overline{\mathbf{\Omega}} = \mathbf{\Omega}_0$.

There are two main properties of the one-dimensional analytic signal that do not extend to the analytic image $z(\mathbf{x})$ generated by $\mathcal{H}_{adj}$. First, if $\mathbf{w}\in\mathbb{C}^n$, then the function $z(\mathbf{w})$ is not holomorphic in general. Second, since the transform $\mathcal{H}_{adj}$ is directional, the analytic image is not unique. Different analytic images are generated by taking the transform in different directions. Each such analytic image associates different amplitude and frequency functions with $s(\mathbf{x})$. However, *all* of these different functions may be obtained from any *one* pair through a straightforward calculation.

4. Examples

We present several discrete examples in this section. The discrete adjusted multidimensional Hilbert transform is constructed by setting the fundamental period of the Fourier transform $\mathcal{F}\{\mathcal{H}_{adj}[s(\mathbf{k})]\}$ equal to $M_{adj}(\omega)S(\omega)$ and extending periodically. The discrete analytic image is then given by $z(\mathbf{k}) = s(\mathbf{k}) + j\mathcal{H}_{adj}[s(\mathbf{k})]$. The synthetic digital image *Diamond* is shown in Fig. 1(a). The imaginary component of $z(\mathbf{k})$ appears in Fig. 1(b). The instantaneous amplitude of $z(\mathbf{k})$ is given in Fig. 1(c), and has been scaled for display. Fig. 1(d) depicts the instantaneous frequency of $z(\mathbf{k})$ as a needle diagram. Similarly, the synthetic image *Radial Chirp* appears in Fig. 1(e). The imaginary component of $z(\mathbf{k})$, instantaneous amplitude, and instantaneous frequency are given in Fig. 1(f), (g), and (h), respectively, where the amplitude has once again been scaled. For single component images, the instantaneous amplitude may be interpreted as *contrast*. As indicated by Fig. 1(g), the contrast of the *Radial Chirp* image is greatest at the center and falls off toward the edges of the image.

For both synthetic images, the amplitude functions exhibit ripples in patterns that tend to be elongated in the direction of action of $\mathcal{H}_{adj}$. The ripples are small compared to the variations in the images. For example, the standard deviation of the amplitude image shown in Fig. 1(c) is a factor of ten smaller than the standard deviation of the image in Fig. 1(a). Note that the frequency vectors in Fig. 1(d) and (h), which are orthogonal to image edges, all lie in the right frequency half-plane. For more complicated natural images, the instantaneous frequency may generally lie in any of the four quadrants.

The image *Wood* appears in Fig. 1(i). $\text{Im}\,[z(\mathbf{k})]$ is given in Fig. 1(j), while the instantaneous amplitude is shown in Fig. 1(k). In this case, the standard deviation of the amplitude is about half as large as that of the image. Hence, if ripples analogous to those seen in Fig. 1(c) and (g) are present, they are obscured by large nonstationary amplitude variations. The instantaneous frequency of $z(\mathbf{k})$ is depicted in Fig. 1(l), where frequency vectors lying in each of the four quadrants are visible.

Finally, the image *Tree* appears in Fig. 1(m). The dominant AM-FM image component was extracted using the *dominant component analysis* (DCA) computational paradigm described in [13], and is shown in Fig. 1(n). The adjusted Hilbert transform of the dominant component is given in Fig. 1(o), while the amplitude and frequency of the dominant component analytic image appear in Fig. 1(p) and (q), respectively. Again, note that the frequency vectors are orthogonal to image edges and that bright areas in the amplitude image correspond to regions of high contrast in the dominant component. Fig. 1(r) and (s) show the log magnitude Fourier spectra of the *Tree* image and the dominant AM-FM component, respectively. The log magnitude Fourier spectrum of the dominant component analytic image, which is supported only in quadrants I and IV, is given in Fig. 1(t).

5. Conclusion

The analytic image is important because it associates unique, physically meaningful amplitude and frequency functions with a real-valued multidimensional signal, thereby providing an unambiguous definition for multidimensional instantaneous frequency. The adjusted Hilbert transform and analytic image are useful in the computation of multi-component AM-FM image models, which have utility in a variety of machine vision processing tasks. We believe that such representations will also find significant future applications in image and video coding for multimedia telecommunications. In discrete computational paradigms where multiband linear filtering is employed to compute multi-component AM-FM models, generation of the analytic image can be incorporated directly into the filters without incurring additional computational overhead.

References

[1] D. Gabor, "Theory of communication", *J. Inst. Elect. Eng. London*, vol. 93, no. III, pp. 429–457, 1946.

[2] F. Peyrin, Y. M. Zhu, and R. Goutte, "Extension of the notion of analytic signal for multidimensional signals. Application to images", in *Signal Processing III: Theories and Applications*, I. T. Young, *et al.*, Ed., pp. 677–680. Elsevier Science Publishers, Amsterdam, B. V. (North-Holland), 1986.

[3] Y. M. Zhu, F. Peyrin, and R. Goutte, "The use of a two-dimensional Hilbert transform for Wigner analysis of 2-dimensional real signals", *Signal Proc.*, vol. 19, pp. 205–220, 1990.

[4] S. L. Hahn, "Multidimensional complex signals with single-orthant spectra", *Proc. IEEE*, vol. 80, pp. 1287–1300, 1992.

[5] J. P. Havlicek, D. S. Harding, and A. C. Bovik, "The mutli-component AM-FM image representation", *IEEE Trans. Image Proc.*, vol. 5, no. 6, pp. 1094–1100, June 1996.

[6] S. Lu and P. C. Doerschuk, "Nonlinear modeling and processing of speech based on sums of AM-FM formant models", *IEEE Trans. Signal Proc.*, vol. 44, no. 4, pp. 773–782, April 1996.

[7] P. Maragos, J. F. Kaiser, and T. F. Quatieri, "Energy separation in signal modulations with applications to speech analysis", *IEEE Trans. Signal Proc.*, vol. 41, no. 10, pp. 3024–3051, October 1993.

[8] P. Maragos and A. C. Bovik, "Image demodulation using multidimensional energy separation", *J. Opt. Soc. Amer. A*, vol. 12, no. 9, pp. 1867–1876, September 1995.

[9] J. Ville, "Théorie et applications de la notation de signal analytique", *Cables et Transmission*, vol. 2A, pp. 61–74, 1948, translated from the French in I. Selin, "Theory and applications of the notion of complex signal," Tech. Rept. T-92, The RAND Corporation, Santa Monica, CA, August 1958.

[10] L. Mandel, "Interpretation of instantaneous frequencies", *Am. J. Phys.*, vol. 42, pp. 840–846, 1974.

[11] L. Cohen and C. Lee, "Instantaneous frequency, its standard deviation and multicomponent signals", *SPIE Adv. Alg. Architectures Signal Proc. III*, vol. 975, pp. 186–208, 1988.

[12] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, NJ, 1970.

[13] J. P. Havlicek, M. S. Pattichis, D. S. Harding, A. C. Christofides, and A. C. Bovik, "AM-FM image analysis techniques", in *Proc. IEEE Southwest Symp. Image Anal., Interp.*, San Antonio, TX, April 8-9, 1996, pp. 195–199.

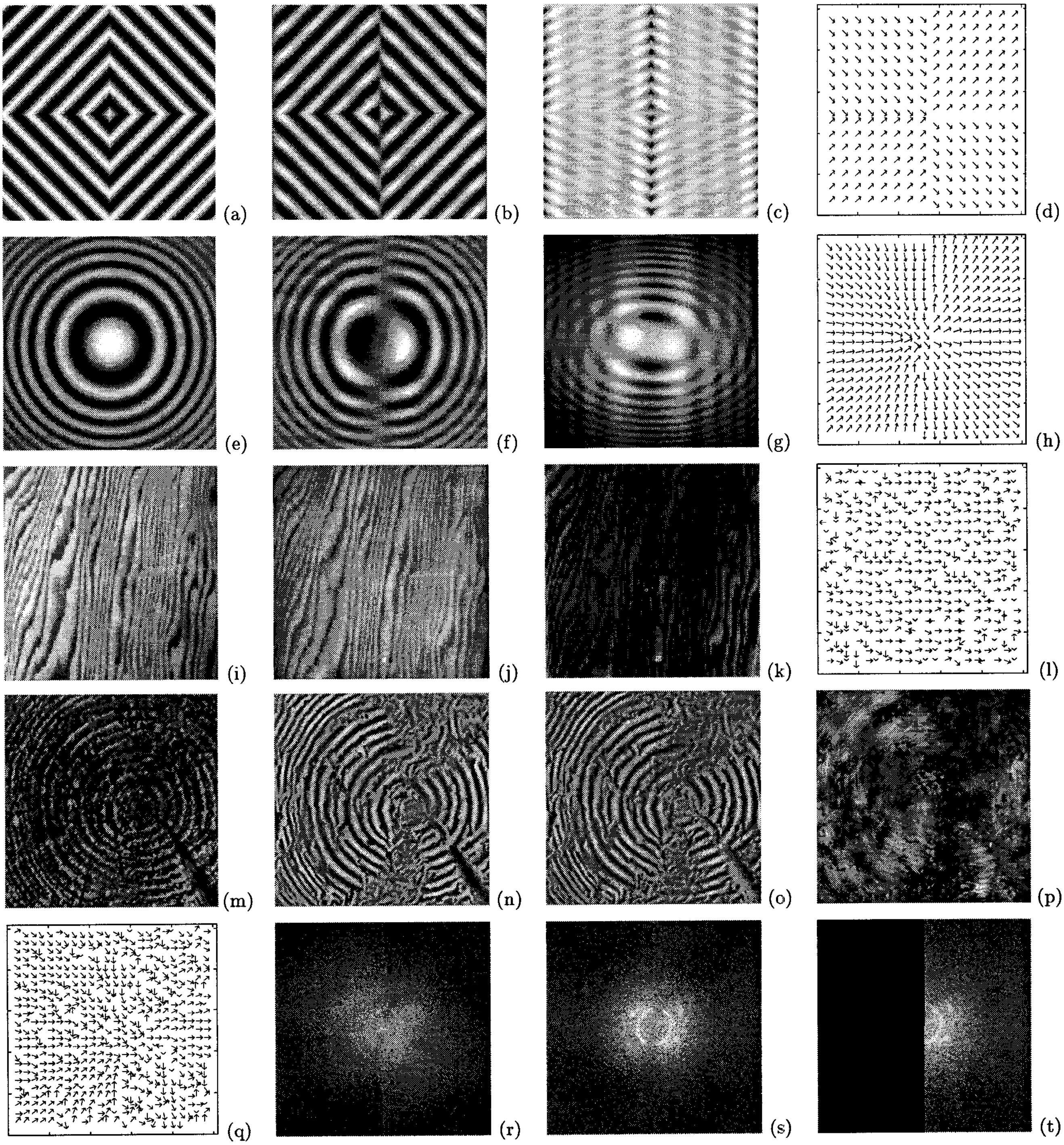

Figure 1. Analytic image examples. The Hilbert transform acts in the horizontal direction. (a) Synthetic image *Diamond.* **(b) Hilbert transform $\mathcal{H}_{adj}$. (c) Instantaneous amplitude. (d) Instantaneous frequency. (e) Synthetic image** *Radial Chirp.* **(f) Hilbert transform $\mathcal{H}_{adj}$. (g) Instantaneous amplitude. (h) Instantaneous frequency. (i)** *Wood* **image. (j) Hilbert transform $\mathcal{H}_{adj}$. (k) Instantaneous amplitude. (l) Instantaneous frequency. (m) Original** *Tree* **image. (n) Dominant component extracted by DCA. (o) Hilbert transform $\mathcal{H}_{adj}$ of dominant component. (p) Instantaneous amplitude of dominant component analytic image. (q) Instantaneous frequency of dominant component analytic image. (r) Log magnitude spectrum of original** *Tree* **image. (s) Log magnitude spectrum of dominant component. (t) Log magnitude spectrum of dominant component analytic image.**