SIGNALS AND SYSTEMS: A CONSISTENT, UNIFIED APPROACH

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Abstract – *This paper addresses two problems associated with the traditional approach to teaching a junior level course in signals and systems. First, students are expected to develop facility with a variety of transforms and we have observed that this overwhelms many, reducing them to simply "memorizing equations." Second, the usual treatment of Dirac's delta as a function not only leads to serious contradictions with the standard calculus, but also leaves intractable schisms surrounding the transforms of important harmonic functions. We present a unified and consistent approach designed to ameliorate these problems using abstract linear algebra and distribution theory. The approach was implemented for four semesters and exit surveys were conducted to assess pedagogical effetiveness.*

Index Terms – Abstract Linear Algebra, Distribution Theory, Fourier Transform, Signals and Systems.

1. INTRODUCTION

In this paper we describe a novel approach to teaching a required junior level signals and systems course in electrical and computer engineering. Several aspects of the standard approach based on [1] concern us and this motivated the study. In this course, students must deal with convolution in discrete and continuous time. They must also learn five transforms, including the Fourier, discrete-time Fourier, Laplace, Z, and DFT.

Our first concern is that students often complain about having to memorize all these "equations." We have observed many who fail to develop a strong intuitive notion of the theory and perceive each transform as being separate and unrelated to the others. Likewise, many students fail to develop intuition about why the output of a linear shift invariant (LSI) system takes the form of a convolution integral or sum. For these students, the two convolutions are merely mysterious equations that must be memorized.

Our second concern is with presentation of the Dirac delta as a mysterious function. This is not only self-inconsistent, but also contradicts freshman and sophomore calculus. For example, the "sifting property" of the Dirac delta states that $\int_{\mathbb{R}} x(t)\delta(t-t_0)\, dt = x(t_0)$. Often, students are given no reason to suspect that this integral is *not* a Riemann integral. While we have an integrand that differs from zero only at a singleton, the value of the integral is generally nonzero — an obvious contradiction with Riemann integration theory.

If the sifting property is accepted on heuristic arguments, as is done often, our students can be persuaded that $\mathcal{F}\{\delta(t)\} = 1$. They are totally ill-equipped, however, to make sense of the companion identity $\mathcal{F}^{-1}\{1\} = \delta(t)$. Even more serious contradictions and confusion arise if they attempt to verify directly the formulas for $\mathcal{F}\{\cos\omega_0 t\}$, $\mathcal{F}\{\sin\omega_0 t\}$, or $\mathcal{F}\{e^{j\omega_0 t}\}$. Similar maladies plague the discrete-time case as well, and this is particularly bothersome.

Our approach is based on injecting two new concepts into the signals and systems course: distribution theory and linear algebra with particular emphasis on the inner product. In a linear algebraic context, the various transforms all reduce to the task of writing a given signal as a linear combination of an appropriate set of basis signals. For students with a firm understanding of inner product there is no need to memorize equations: all of the transforms can be derived and computed by following a single unified and consistent procedure, of which distribution theory is a key element in cases like $\delta(t)$ and the harmonic signals mentioned above. Using this same procedure, it is almost obvious that the response of an LSI system takes the form of a convolution.

The main argument that we have heard against the approach we describe is that the involved mathematics is too advanced for engineering undergraduates, and we set out to test this hypothesis. Our approach is sketched in Section 2, where we focus on only the Fourier and discrete-time Fourier transforms in the interest of brevity. Results of the exit surveys we conducted for assessment are briefly presented in Section 3, while conclusions appear in Section 4.

2. THE APPROACH

In this section we briefly outline our approach. We denote the inner product between functions f and g by $\langle f, g\rangle$. We find that our undergraduates are generally comfortable and facile with the Euclidean inner product, or *dot product* in $\mathbb{R}^3$. They readily recall that an arbi-

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trary vector $\vec{v} \in \mathbb{R}^3$ may be written as $\vec{v} = c_1\vec{\imath} + c_2\vec{\jmath} + c_3\vec{k}$, where $c_1 = \langle \vec{v}, \vec{\imath} \rangle$, $c_2 = \langle \vec{v}, \vec{\jmath} \rangle$, and $c_3 = \langle \vec{v}, \vec{k} \rangle$. While computation of the inner product in more sophisticated spaces is similar, one of the two vectors must be conjugated if complex values are involved. Our convention is to always conjugate the second vector.

2.1 . Discrete-Time Time Domain Analysis

Consider the Kronecker delta $\delta[n]$, which is straightforward to graph. By substituting the ordinate values for the "stems" in the usual graph [2], surrounding these numbers above and below by square brackets, and turning the resulting vector up on its "side", we rapidly convince our students that $\delta[n]$ in $\mathbb{C}^\infty$ is analogous to $\vec{\imath}$ in $\mathbb{R}^3$. They are then able to grasp the correspondence between $\{\vec{\imath}, \vec{\jmath}, \vec{k}\}$ in $\mathbb{R}^3$ and the set of integer translates $\{\delta[n-k]\}_{k\in\mathbb{Z}}$ in a countably infinite dimensional space of signals $x[n]$.

By the same procedure that was used in $\mathbb{R}^3$, we write

$$x[n] = \sum_{k\in\mathbb{Z}} c_k \delta[n-k], \tag{1}$$

where

$$c_k = \Big\langle x[n], \delta[n-k] \Big\rangle = \sum_{n\in\mathbb{Z}} x[n]\delta[n-k] = x[k]. \tag{2}$$

We emphasize to our students that the usual graph of $x[n]$ may be interpreted as a depiction of the coefficients c_k in (1) that weight each shifted Kronecker delta.

For an LSI system H with impulse response $h[n]$, we substitute (2) into (1) to establish that the system response is given by convolution:

$$\begin{aligned} y[n] &= H\{x[n]\} = H\left\{\sum_{k\in\mathbb{Z}} x[k]\delta[n-k]\right\} && (3)\\ &= \sum_{k\in\mathbb{Z}} x[k] H\{\delta[n-k]\} && (4)\\ &= \sum_{k\in\mathbb{Z}} x[k]h[n-k] = x[n] * h[n]. && (5) \end{aligned}$$

While (3)-(5) convince our students on an intellectual level, they typically remain uncomfortable with the concept until a deeper intuition is developed.

To achieve this, we consider a specific causal example where the input $x[n]$ and impulse response $h[n]$ are both short, finitely supported signals. We write $x[n]$ according to (1) and, for some specific time like $n = 10$, consider the response to each input term. This leads to the fact that, at $n = 10$, the system responds not only to the input term that arrived at $n = 10$, but also to input terms that arrived earlier. For example, input term $x[8]\delta[n-8]$ arrived at $n = 8$ and caused the response $x[8]h[n-8]$ to begin coming out of the system at that time; by time $n = 10$, the contribution of this input term to $y[10]$ is $x[8]h[2]$. The term $x[9]h[1]$ is also present in $y[10]$, demonstrating that *earlier inputs* corresponding to $x[n]$ for *smaller* n contribute *later* $h[k]$ with *larger* k; this explains why the indices of $x[k]$ and $h[n-k]$ go in "different directions" in the sum of (5). We obtain the desired pedagogical result by relaxing the causality and finite support assumptions and realizing that there was nothing special about the time $n = 10$. By an intuitively satisfying path, this leads back to the expression (5) for $y[n]$. Following [1], we establish the commutivity of discrete convolution by making a straightforward change of variable.

2.2 . Distributions

We introduce distribution theory on the Schwartz class following [3], and we do not concern ourselves with details of how the theory is extended to larger signal classes [3,4]. Before doing this, however, we briefly cover the modern integration theory that will be required later in applying the Riemann-Lebesgue lemma (RLL) [4–6]. We outline the computational aspects of Lebesgue measure on the line and bring our students to a point where they can integrate simple "step functions" with countable ranges. We then explain qualitatively how the monotone and dominated convergence theorems are used to integrate more general functions that might have uncountable ranges [7,8]. Finally, we state without proof the fact that any Riemann integrable function is also Lebesgue integrable and that the two integrals are equal when they both exist.

We begin our discussion of distributions by observing that there are situations in which the effect that a signal has on a system or the way in which it interacts with other signals through the inner product are significant, and yet the particular values that the signal takes at particular times are of no interest. A familiar example is when the oscilloscope is used to observe the input and output of a circuit driven by short pulses. If we view the signals at a large time scale compared to the pulse durations, then all input pulses look like vertical lines on the scope even though they may have distinct shapes when viewed at a finer scale. In such cases we would like to have a single mathematical object capable of modeling whole classes of signals that are not perceptibly different from one another when viewed at the scale of interest. Distributions provide this capability, whereas ordinary functions cannot.

Formally, a distribution f is a continuous linear functional defined on an appropriate signal space: f maps each signal to a number. Given a locally integrable function $f(t)$, one easy way to construct a distribution is by inner product with f:

$$\langle x(t), f(t)\rangle = \int_{\mathbb{R}} x(t)f^*(t)\, dt, \tag{6}$$

where superscript asterisk denotes conjugation. Distributions such as (6) are called *regular*. We alternatively use the symbols f, $\langle\cdot, f(t)\rangle$, and $\langle x(t), f(t)\rangle$ for the distribution (6).

A *singular* distribution $\langle\cdot, g(t)\rangle$ is a continuous linear functional for which there does not exist a *function* $g(t)$ satisfying $\langle\cdot, g(t)\rangle = \int_{\mathbb{R}} x(t)g^*(t)dt$. The prototypical example of a singular distribution is the Dirac delta

$$\langle x(t), \delta(t)\rangle = x(0). \tag{7}$$

Singular distributions should be viewed as a generalization of the notion of inner product that frees us from specifying the precise values of the signal $g(t)$ in (7) at every time. To maintain notational consistency between regular and singular distributions, we write the *symbolic* integral

$$\int_{\mathbb{R}} x(t)\delta(t)\, dt = x(0) \tag{8}$$

for $\langle x(t), \delta(t)\rangle$, where it is understood that no ordinary *function* $\delta(t)$ exists satisfying (8) in the sense of Riemann or Lebesgue integration. To paraphrase Zemanian [3], "the left side of (8) has *no meaning* other than that given to it by the right side." Another reason for writing the symbolic integral in (8) is that symbolic manipulations such as variable changes to accommodate time shifting and time scaling of distributions, if carried out as though there were an actual integral in (8), lead to consistent definitions for these operations over the class of singular distributions.

Two distributions $f(t)$ and $g(t)$ are said to be *equal*, or *equal in the sense of distributions*, if $\langle x(t), f(t)\rangle = \langle x(t), g(t)\rangle$ for all signals $x(t)$ in the space of interest. Thus, f and g are equal if they both map every signal to the same number.

To define operations on distributions, we first use ordinary integration to determine the behavior of the operation on regular distributions. We then *define* the behavior on singular distributions to be consistent with that on regular distributions. Consider time translation as an example. Given the regular distribution $f(t)$, we wish to define the distribution $f(t-t_0)$. Making a straightforward change of variable in (6), we have

$$\begin{aligned} \langle x(t), f(t-t_0)\rangle &= \int_{\mathbb{R}} x(t)f^*(t-t_0)\, dt && (9)\\ &= \int_{\mathbb{R}} x(t+t_0)f^*(t)\, dt && (10)\\ &= \langle x(t+t_0), f(t)\rangle. && (11)\end{aligned}$$

Based on (11), we *define* the time-translated distribution

$$\langle x(t), g(t-t_0)\rangle = \langle x(t+t_0), g(t)\rangle \tag{12}$$

for *all* distributions $g(t)$, both regular and singular.

The theme seen in (12), where an operation on a distribution is defined by moving an equivalent operation onto the signal $x(t)$, recurs again and again. By the same type of reasoning, we define the following:

1. *Addition:* $\langle x(t), f(t)+g(t)\rangle = \langle x(t), f(t)\rangle + \langle x(t), g(t)\rangle$.
2. *Scalar multiplication:* $\langle x(t), \alpha f(t)\rangle = \langle \alpha^* x(t), f(t)\rangle$.
3. *Time scaling:* $\langle x(t), f(at)\rangle = \langle \frac{1}{|a|}x\left(\frac{t}{a}\right), f(t)\rangle$. Note that this easily establishes that, in the sense of distributions, $\langle x(t), \delta(at)\rangle = \langle \frac{1}{|a|}x(\frac{t}{a}), \delta(t)\rangle = \frac{1}{|a|}x(0)$, implying that $\delta(at) = \frac{1}{|a|}\delta(t)$.
4. *Time differentiation:* $\langle x(t), f'(t)\rangle = \langle -x'(t), f(t)\rangle$. This may be used to rigorously establish that, for the unit step function $u(t)$, $u'(t) = \delta(t)$.

2.3 . Riemann-Lebesgue Lemma

The signal $e^{j\omega t}$ is fundamental in many engineering disciplines. Yet we are unlikely to encounter this signal in engineering practice, since its real and imaginary parts oscillate periodically for *all time*. However, it is not uncommon to encounter a signal $x(t)$ that behaves like $e^{j\omega t}$ for a time period that is much longer than the interval of interest for the problem at hand. In such cases, it seems reasonable to model $x(t)$ as being equal to $e^{j\omega t}$ and not worry about the fact that this model is inaccurate for distant past and future times. This motivates the consideration of $e^{j\omega t}$ as a distribution rather than a function: it is the signal's behavior in systems and inner products over the time interval of interest for the current problem, and not the specific values of the signal at *all* times, that is significant.

The Riemann-Lebesgue lemma (RLL) is concerned with the distribution $e^{j\omega t}$ in the limit as $\omega \to \infty$. The RLL states that, in the sense of distributions, $\lim_{\omega\to\pm\infty} e^{j\omega t} = 0$. This limiting distribution maps every signal $x(t)$ to the same number as does the regular distribution 0, namely zero. The most important consequence of the RLL is the fact that

$$\lim_{A\to\infty} \frac{\sin At}{\pi t} = \delta(t). \tag{13}$$

Our rigorous proof of (13), which has proven to be both accessible and satisfying to our students despite its use of the dominated convergence theorem, is omitted here in the interest of brevity.

2.4 . Continuous-Time Time Domain Analysis

With the mathematical machinery we now have in place, time domain analysis of continuous-time signals and systems is no different than it was in discrete time. The basis of interest is $\{\delta(t-\tau)\}_{\tau\in\mathbb{R}}$, the set of translates of the Dirac delta. We write the signal $x(t)$ as an (uncountable) linear composition of the basis signals according to

$$x(t) = \int_{\mathbb{R}} c_\tau \delta(t-\tau)\, d\tau. \qquad (14)$$

As before, the coefficients in (14) are nothing more than inner products between $x(t)$ and the respective basis signals:

$$c_\tau = \langle x(u), \delta(u-\tau)\rangle = x(\tau). \qquad (15)$$

The expression (14) for $x(t)$ then reduces to

$$x(t) = \langle x(\tau), \delta(t-\tau)\rangle. \qquad (16)$$

For an LSI system H with impulse response $h(t)$, computation of the response $y(t)$ to input $x(t)$ is then straightforward:

$$\begin{aligned} y(t) &= H\{x(t)\} = H\{\langle x(\tau), \delta(t-\tau)\rangle\} && (17)\\ &= \langle x(\tau), H\{\delta(t-\tau)\}\rangle && (18)\\ &= \langle x(\tau), h(t-\tau)\rangle && (19)\\ &= \int_{\mathbb{R}} x(\tau)h(t-\tau)\, d\tau = x(t) * h(t). && (20) \end{aligned}$$

Note that the operation in (20) is ordinary integration: the action of the system H transforms the singular distribution (16) into a regular distribution (20) that may be interpreted as an ordinary Riemann or Lebesgue integral, provided that $h(t)$ is locally integrable. To develop an intuitive notion of continuous-time convolution, we proceed along the same lines as the discussion outlined in Section 2.1 for the discrete-time case.

2.5 . Discrete-Time Frequency Analysis

For discrete-time Fourier analysis, the basis of interest is $\{e^{j\omega n}\}_{\omega\in[-\pi,\pi)}$. We begin by demonstrating that the signal $x[n] = e^{j\omega n}$ is an eigenfunction of any LSI system. With $h[n]$ the system impulse response, we have

$$\begin{aligned} y[n] &= \sum_{k\in\mathbb{Z}} e^{j\omega(n-k)}h[k] && (21)\\ &= e^{j\omega n}\sum_{k\in\mathbb{Z}} e^{-j\omega k}h[k] && (22)\\ &= \lambda x[n]. && (23) \end{aligned}$$

Thus, if we write an arbitrary signal $x[n]$ as a linear composition of the basis signals $\{e^{j\omega n}\}_{\omega\in[-\pi,\pi)}$, it is particularly easy to compute the system response to each input term in this composition. We conveniently write the collection of all eigenvalues λ in (23) for all signals in the basis $\{e^{j\omega n}\}_{\omega\in[-\pi,\pi)}$ together as a function $H(e^{j\omega})$, $-\pi \le \omega < \pi$. This collection of eigenvalues, or function, is known as the *frequency response* of the system H.

The procedure for writing an arbitrary signal $x[n]$ in terms of the basis is no different than it was in Sections 2.1 and 2.4. We have

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} c_\omega e^{j\omega n}\, d\omega, \qquad (24)$$

were the leading factor $\frac{1}{2\pi}$ arises because the Fourier transform is not a true Hilbert space isomorphism when the frequency variable is expressed in radian units. Intuitively, this is equivalent to saying that the basis $\{e^{j\omega n}\}_{\omega\in[-\pi,\pi)}$ is not orthonormal. The factor $\frac{1}{2\pi}$ can be eliminated from (24) by using the basis $\{e^{j2\pi fn}\}_{f\in[-1/2,1/2)}$ where the frequency variable f is expressed in Hertz; we use the radian frequency basis for consistency with [1, 2, 9].

The coefficients c_ω in (24) are computed as inner products between the signal $x[n]$ and the respective basis signals as usual:

$$c_\omega = \langle x[n], e^{j\omega n}\rangle = \sum_{k\in\mathbb{Z}} x[n]e^{-j\omega n}. \qquad (25)$$

For convenience, we write the coefficients c_ω for all of the basis signals together as a function $X(e^{j\omega})$, whereupon (24) and (25) become the familiar discrete-time Fourier transform equations

$$\begin{aligned} X(e^{j\omega}) &= \sum_{k\in\mathbb{Z}} x[n]e^{-j\omega n}, && (26)\\ x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\, d\omega. && (27) \end{aligned}$$

We now observe from (22) that the eigenvalue $\lambda = H(e^{j\omega})$ in (23) is precisely the Fourier transform of $h[n]$ evaluated at ω; *i.e.*, that the frequency response of the system is given by $H(e^{j\omega}) = \mathcal{F}\{h[n]\}$. Using the expression (27) for the input signal $x[n]$, we compute the system output according to

$$\begin{aligned} y[n] &= H\{x[n]\} && (28)\\ &= H\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\, d\omega\right\} && (29)\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})H\left\{e^{j\omega n}\right\} d\omega && (30)\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})H(e^{j\omega})e^{j\omega n}\, d\omega && (31)\\ &= \mathcal{F}^{-1}\{X(e^{j\omega})H(e^{j\omega})\}, && (32) \end{aligned}$$

which provides an intuitive understanding of the fact that $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$ and establishes that writing the signals $x[n]$, $h[n]$, and $y[n]$ in terms of the basis $\{e^{j\omega n}\}_{\omega\in[-\pi,\pi)}$ instead of the basis $\{\delta[n-k]\}_{k\in\mathbb{Z}}$ results in pointwise multiplication for the system output rather than convolution.

To conclude this section, we consider the Fourier transform pair

$$e^{j\omega_0 n} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi \sum_{k\in\mathbb{Z}} \delta(\omega - \omega_0 - 2\pi k). \tag{33}$$

Using (7) and (27), it is easy to establish the inverse transform in (33). For the forward transform, we apply several routine trigonometric substitutions to obtain

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n\in\mathbb{Z}} e^{j\omega_0 n} e^{j\omega n} = \lim_{M\to\infty} \sum_{n=-M}^{M} e^{j(\omega_0-\omega)n} \\ &= \lim_{M\to\infty} \frac{\sin[(\omega-\omega_0)(M+\frac{1}{2})]}{\sin[\frac{1}{2}(\omega-\omega_0)]}. \end{aligned} \tag{34}$$

Application of the RLL and L'Hôpital's rule to (34) yields $2\pi\delta(\omega-\omega_0)$ for the fundamental period of $X(e^{j\omega})$. Since the Fourier transform sum is not convergent in the ordinary sense in this case, the intrinsic 2π periodicity of $X(e^{j\omega})$ is not preserved in the passage through the generalized (distributional) limit however. Although the frequency support of the basis $\{e^{j\omega n}\}_{\omega\in[-\pi,\pi)}$ implied by (27) coincides with the fundamental period of $X(e^{j\omega})$, the consequence of an intrinsically periodic spectrum is that the transform can be inverted using a basis that covers *any* connected frequency interval of length 2π. For consistency it is therefore necessary to replicate the fundamental period $2\pi\delta(\omega-\omega_0)$ obtained from (34), which results in the right side of (33).

2.6. Continuous-Time Frequency Analysis

Analogous to the discrete-time case, the basis of interest for continuous-time frequency analysis is $\{e^{j\omega t}\}_{\omega\in\mathbb{R}}$. For any fixed ω and any LSI system H with impulse response $h(t)$, the response to the eigenfunction $e^{j\omega t}$ is

$$y(t) = e^{j\omega t} * h(t) = \lambda e^{j\omega t}, \tag{35}$$

where the associated eigenvalue is given by

$$\lambda = \int_{\mathbb{R}} h(t) e^{-j\omega t}\, dt. \tag{36}$$

We define the system frequency response $H(\omega)$ as a map from each $\omega \in \mathbb{R}$ to the eigenvalue associated with the basis signal $e^{j\omega t}$.

For an arbitrary signal $x(t)$ that is to be written as a linear composition of the basis, the required coefficients are given as before by the inner product

$$X(\omega) = \langle x(t), e^{j\omega t}\rangle = \int_{\mathbb{R}} x(t) e^{-j\omega t}\, dt. \tag{37}$$

Summing up these coefficients times their respective basis signals as before and multiplying by a constant to account for the fact that the basis is not orthonormal then yields

$$x(t) = \frac{1}{2\pi} \int_{\mathbb{R}} X(\omega) e^{j\omega t}\, d\omega. \tag{38}$$

Eq. (37) defines the continuous-time Fourier transform, while the inverse Fourier transform is defined by (38). From (36), it is clear that the system frequency response is given by $H(\omega) = \mathcal{F}\{h(t)\}$.

For an arbitrary input $x(t)$ written according to (38), the system output is given by

$$\begin{aligned} y(t) &= H\{x(t)\} = H\left\{\frac{1}{2\pi}\int_{\mathbb{R}} X(\omega) e^{j\omega t}\, d\omega\right\} && (39) \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} X(\omega) H\{e^{j\omega t}\}\, d\omega && (40) \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} X(\omega) H(\omega) e^{j\omega t}\, d\omega && (41) \\ &= \mathcal{F}^{-1}\{X(\omega)H(\omega)\}. && (42) \end{aligned}$$

Like (32), (42) shows that if $x(t)$, $h(t)$, and $y(t)$ are written in terms of the spectral basis instead of the Dirac basis, then the system output is given by pointwise spectral multiplication as opposed to convolution.

Applying the distribution theory developed in Sections 2.2 and 2.3, several important results become extremely easy to establish. For example,

$$\begin{aligned} \mathcal{F}\{\delta(t)\} &= \int_{\mathbb{R}} \delta(t) e^{-j\omega t}\, dt && (43) \\ &= \langle e^{-j\omega t}, \delta(t)\rangle = 1. && (44) \end{aligned}$$

The corresponding inverse transform is

$$\begin{aligned} \mathcal{F}^{-1}\{1\} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}\, d\omega && (45) \\ &= \frac{1}{2\pi}\lim_{A\to\infty}\int_{-A}^{A} \cos\omega t\, d\omega && (46) \\ &= \lim_{A\to\infty} \frac{\sin At}{\pi t} = \delta(t). && (47) \end{aligned}$$

The transform pair $e^{j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\delta(\omega-\omega_0)$ may then be established by applying the Fourier transform frequency shift property to (44) and (47) or directly using distribution theory and arguments similar to those leading to (34)

3. RESULTS

The unified approach described in Section 2 was implemented in our junior level signals and systems course over four consecutive semesters. During the last two semesters, we conducted exit surveys of the students to assess the effectiveness of the approach. The survey consisted of a series of six questions; two focused on the use of linear algebra to teach forward and inverse transforms and four focused on the use of distribution theory to teach the Dirac delta. Student responses to questions regarding the use of linear algebra, in particular the inner product, to teach transforms were very favorable in terms of helping them learn the material. Out of 60 total respondents, 44 said that the linear algebra helped them, five said that it hurt them, and 11 said that it had no effect. Forty-two students said that the transforms should be taught in terms of inner product in the future, while 15 said that they should be taught using the traditional approach and three had no opinion.

Student responses to questions regarding the helpfulness of the distribution theory were mixed and far less conclusive. The responses revealed that while the students found the topic interesting, a significant number of them had difficulty understanding the distribution theory and indicated a preference for the traditional method of introducing the Dirac delta "function." Specifically, 42 students said that they found the theory interesting, while 15 said that they did not (three students did not respond to this question). Thirty students said that the distribution theory helped them, five said that it hurt them, 24 said that it had no effect, and one student did not respond. In response to our question of whether the Dirac delta should be taught using distribution theory or taught in the conventional way as a "function," 21 students favored the distribution theory, 27 recommended the conventional approach, and 12 either didn't know or elected not to respond.

4. CONCLUSION

While the scope of our study was too limited to support definitive conclusions, the results strongly suggest that teaching signals and systems from a linear algebraic viewpoint is beneficial to student learning. In particular, unified explicit treatment of the forward transforms as inner products and the inverse transforms as linear compositions of appropriate basis signals seems to increase the students' intuitive comprehension of the theory while concomitantly reducing the need to "memorize equations." Moreover, this approach powerfully leverages the students' thorough understanding of "dot product" and basis in $\mathbb{R}^3$.

The results also suggest that a significant portion of the students were able to understand the distribution theory in our approach and also found it helpful. However, only about one-third of them recommended that distributions should be taught in the future, whereas nearly half said that the conventional approach should be used for future semesters. One plausible explanation that we have entertained for these mixed results is that, with the addition of the new topics, the course simply contains too much material for a single semester. While the introduction of linear algebraic and distributional concepts could be spread out over several math and engineering courses to address this issue, doing so would require significant curricular revisions. Finally, we feel that our results cast significant doubt on the idea that distribution theory, abstract linear algebra, and modern integration theory are topics that are too advanced for engineering undergraduates.

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