THE OPTIMAL SOLUTIONS TO THE CONTINUOUS- AND DISCRETE-TIME VERSIONS OF THE HIRSCHMAN UNCERTAINTY PRINCIPLE

V. DeBrunner[1], M. Özaydın[2], T. Przebinda[2] and J. Havlicek[1]

The University of Oklahoma
[1]School of Electrical and Computer Engineering
[2]Department of Mathematics
Norman, OK 73019, USA

ABSTRACT

We have previously developed an uncertainty measure that is suitable for finitely-supported (N samples) discrete-time signals. A specific instance of our measure has been termed the "Discrete Hirschman Uncertainty Principal" in the literature, and we have adopted this terminology for our more general measure. We compare the optimal signals of this discrete version to the already determined optimal signals of the (continuous-time) Hirschman Uncertainty Principal. From our comparison, we conclude that a basic premise in signal processing, that if we sample densely enough, the discrete-time case directly corresponds to the continuous-time case, is not correct in this instance. The arithmetic of N, which seems to have no analog in continuous time, is crucial to the construction of the Hirschman optimal discrete representation. We suggest that more work in this important area be performed to determine what impact this has, and to find out how widespread this problem may be.

1. INTRODUCTION

In [1], we introduced a measure H_p that predicts the compactness of a discrete-time signal in the sample-frequency phase plane. This measure was used to show that discretized Gaussian pulses may not be the most compact basis, and a lower limit on the compaction of the phase plane was conjectured. We have since discovered that part of this conjectured lower limit was proven in [2] under the moniker "a discrete Hirschman's uncertainty principle." However, that result did not describe the characteristics of the signals that meet the limit, as our conjecture did [3]. We further argued in [4] that this measure indicates two possible "best basis" options:

1. The multi-transform (non-orthogonal) option
2. The orthogonal discrete Hirschman uncertainty principle option

We have discussed many results in the first option (see [1] for many references to this work). We have very recently discovered a result for the second option [5]. We first discuss this result, giving the theorem that defines signals that meet the bound and their uniqueness. We will then briefly review the continuous-time Hirschman uncertainty principle, and discuss the results that previous researchers have found regarding optimal signals in this case. From these results, we conclude that the conditions of finite support and discrete time sampling significantly alter the solution, and that no matter how densely we sample, the discrete-time result does not directly correspond to the continuous-time solution. We acknowledge that our example given in this paper does not address these other two important questions: Is the difference relevant? Is the difference universal? These two questions suggest that more work in this important area needs to be performed to determine the impact as well as the applicability of this "non-convergence" of the discrete-time case to the continuous-time case. However, our work clearly indicates that treating the discrete-time case as a continuum of the continuous-time case will not produce our Hirschman optimal discrete representation.

2. THE DISCRETE HIRSCHMAN UNCERTAINTY PRINCIPLE

2.1 Definitions

Fix a positive integer N. Let A denote the ring $\mathbb{Z}/N\mathbb{Z}$. Thus $A=\{0,1,2,\ldots,N-1\}$, with the addition and multiplication modulo N. Often we shall view A as a group with respect to addition.

The Heisenberg group of degree one, with coefficients in A, is the group $G_1(A)$ of all matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} (x,y,z\in A)$$

We shall identify $G_1(A)$ with the cartesian product $A\times A\times A$ via the map

$$G_1(A)\ni\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}\rightarrow(x,y,z)\in A\times A\times A \qquad (1)$$

In terms of (1) the matrix multiplication and the inverse look as follows:

$$(x,y,z)(x',y',z')=(x+x',y+y',z+z'+xy'),$$
$$(x,y,z)^{-1}=(-x,-y,-z+xy) \quad (x,y,z,x',y',z'\in A)$$

Let

$$\mathcal{X}(a)=e^{j\frac{2\pi a}{N}} \quad (a\in A)$$

This is a unitary character of the (additive) group A. For a function $u:A\to\mathbb{C}$ let

$$\|u\|_2=\left(\sum_{a\in A}|u(a)|^2\right)^{\frac{1}{2}} \tag{2}$$

and let $L^2(A)$ denote the Hilbert space of all such functions, with the norm (2). Let

$$\rho(x,y,z)u(a)=\mathcal{X}(ay+z)u(a+x) \quad \left(u\in L^2(A);a,x,y,z\in A\right) \tag{3}$$

It is easy to check that ρ is a group homomorphism from $G_1(A)$ to the group of unitary operators on $L_2(A)$. In other words, ρ is a unitary representation of $G_1(A)$ on the space $L_2(A)$.

Recall the discrete Fourier transform, defined with respect to the character $\mathcal{X}$:

$$\mathcal{F}u(b)=\hat{u}(b)=|A|^{-\frac{1}{2}}\sum_{a\in A}u(a)\mathcal{X}(-ab) \quad \left(u\in L^2(A),b\in A\right)$$

Here $|A|=N$ is the cardinality of the set A. The inverse Fourier transform is given by

$$u(a)=|A|^{-\frac{1}{2}}\sum_{b\in A}\hat{u}(b)\mathcal{X}(ab) \quad \left(u\in L^2(A),a\in A\right)$$

A straightforward calculation shows that

$$\mathcal{F}\rho(x,y,z)\mathcal{F}^{-1}=\rho(-y,x,z-xy) \quad (x,y,z\in A) \tag{4}$$

In other words, the Fourier transform normalizes the group $\rho(G_1(A))$. For $u\in L^2(A)$, with $\|u\|_2=1$, let

$$H(u)=-\sum_{a\in A}|u(a)|^2\log\left(|u(a)|^2\right)$$

and let

$$H_p(u)=pH(u)+(1-p)H(\hat{u})$$

It is easy to see that

$$H_p(\rho(h)u)=H_p(u) \quad \left(h\in G_1(a),0\le p\le 1\right)$$

We would like to consider $u\in L^2(A)$ with $\|u\|=1$ equivalent to $v=\lambda u$ where $|\lambda|=1$. As $H(u)=H(v)$ and $H_p(u)=H_p(v)$ for equivalent u and v, H and H_p are defined on the equivalence classes. This set of equivalence classes form a complex projective space which we will denote by $P(A)$. Note that being orthogonal is well-defined on the equivalence classes, so a subset of $P(A)$, being orthonormal makes sense. There is an induced action of the Heisenberg group $G_1(A)$ on $P(A)$ defined via (3) at the level of representatives for the equivalence classes. Below we will use the same symbol u for an element of $L^2(A)$ with $\|u\|=1$ and the equivalence class it represents in $P(A)$.

If B is a subset of A, let $\mathbb{I}_B$ denote the indicator function of B. Thus $\mathbb{I}_B(a)=1$ if $a\in B$, and $\mathbb{I}_B(a)=0$ if $a\in A\setminus B$.

2.2 Theorem

Here is our main theorem.

Theorem 1 (Main Theorem)

(a) *If $u\in P(A)$, then $H_{\frac{1}{2}}(p)\ge\frac{1}{2}\log(|A|)$*

(b) *The set of vectors $u\in P(A)$ and $H_{\frac{1}{2}}(u)=\frac{1}{2}\log(|A|)$ coincides with the union of the orbits*

$$\rho(G_1(A))\frac{1}{\sqrt{|B|}}\mathbb{I}_B \quad (B\text{ - a subgroup of }A) \tag{5}$$

(c) *Each orbit (5) is an orthonormal basis of $L^2(A)$.*

(d) *The set of vectors $u\in P(A)$ and $H_p(u)=\frac{1}{2}\log(|A|)$ for all $0\le p\le 1$ is non-empty if and only if $|A|$ is a square. In this case, this set coincides with the orbit (5) for the unique subgroup $B\subseteq A$ of cardinality $|B|=\sqrt{|A|}$.*

Part (a) of the above theorem has been proven by A. Dembo, T. M. Cover and J. A. Thomas in [2]. The idea of their proof is based on Hirschman's work in [6]. In fact, those authors name the inequality (a) "the Hirschman Uncertainty Principle." Following this line we have chosen the title of this paper. While unaware of the work in [2], we conjectured a result close to the above theorem in [1]. Part (c) suggests a close connection of the functions listed in (b) with wavelets, along the lines explored partially in [7].

A complete proof of the theorem is given in [5]. The importance in this paper is that we now not only have a signal that meets the lower bound, but that we have uniquely identified all signals that meet the lower bound, and determined that they can form an orthonomal basis. From this theorem, we can define the Hirschman Optimal Transform (HOT) that may be computed via a fast algorithm due to its relationship to the Fourier transform.

2.3 The HOT

The basis functions that define the HOT are derived according to the second item of the theorem. The functions are suggested in [12], though in that paper they were derived without any measure using entropy. Consequently, we use the K-dimensional discrete Fourier transform (DFT) as the originator functions for an $N = K^2$-dimensional HOT basis. Each of the basis functions from the K-dimensional DFT are shifted and interpolated to produce the sufficient N basis functions that define the HOT. We note that the DFT basis could be extended in a similar manner to produce a $N = KL$-dimensional transform, but as the fourth item of the theorem shows, this transform is not HOT optimal. Plots of the $N = 4$ case are shown in Figure 1.

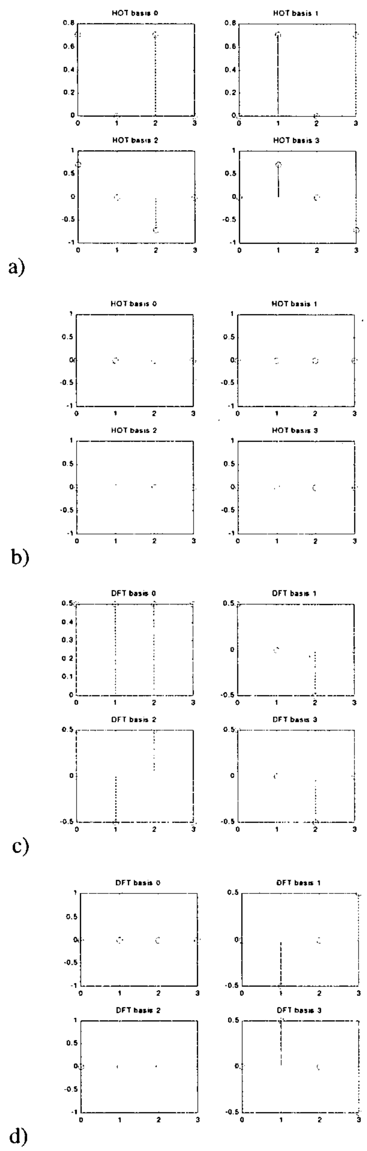


Figure 1. HOT and DFT basis functions: a) The real part of the HOT basis functions b) The imaginary part of the HOT basis functions c) The real part of the DFT basis functions d) The imaginary part of the DFT basis functions

Because the HOT is based on periodic shifts of the DFT, the $N = K^2$-point HOT can be accomplished using K separate K-point DFT computations. However, since the HOT requires lengths $N = K^2$, the efficiency of any computational procedure will depend on the exact length N. For $N = 4, 16, 64, 256$ etc., this provides a fast HOT that requires $O(N \log K)$ computations. For other lengths the efficiency of the HOT calculation will be less. Thus, in general, the N-point HOT is more efficient than the N-point DFT.

3. THE CONTINUOUS-TIME HIRSCHMAN UNCERTAINTY PRINCIPLE

3.1 The Hirschman Uncertainty Principle

First, we provide some definitions similar to the discrete-time case. Let $S(\mathbb{R}^n)$ be the Schwartz space of functions on the Euclidian space $\mathbb{R}^n$ [8]. Recall the Fourier transform

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi\xi \cdot x} dx, \quad \left(f \in S(\mathbb{R}^n), \xi \in \mathbb{R}^n\right)$$

where $\xi \cdot x = x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n$. As in (2), let

$$\|f\|_2 = \left(\int_{\mathbb{R}^n} |f(x)|^2 dx\right)^{\frac{1}{2}}, \quad \left(f \in S(\mathbb{R}^n)\right)$$

Consider a function $f \in S(\mathbb{R}^n)$ with $\|f\|_2 = 1$. Then $|f(x)|^2$ is a probability distribution on $\mathbb{R}^n$. Hence the notion of entropy, introduced by Shannon [9] applies to $|f(x)|^2$. We shall denote by $H(f)$ the entropy of $|f(x)|^2$:

$$H(f) = -\int_{\mathbb{R}^n} |f(x)|^2 \log\left(|f(x)|^2\right) dx, \quad \left(f \in S(\mathbb{R}^n), \|f\|_2 = 1\right)$$

As is well known, $\|f\|_2 = \|\hat{f}\|_2$. Hence, as in the previous section, we define

$$H_p(f) = pH(f) + (1-p)H(\hat{f}), \quad \left(f \in S(\mathbb{R}^n), \|f\|_2 = 1\right)$$

The following theorem, conjectured by Hirshman [6], has been proven by Beckner [10],[11]:

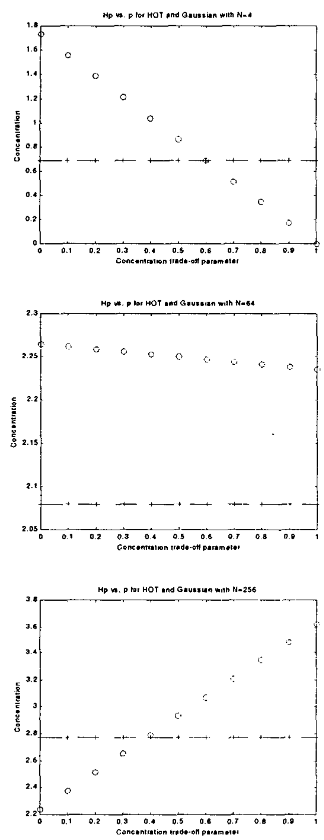
Theorem. *Let* $f \in S(\mathbb{R}^n), \|f\|_2 = 1$. *Then*

$$H_{\frac{1}{2}}(f) \geq \frac{n}{2} \ln\left(\frac{e}{2}\right)$$

This is a beautiful result, and we would like to thank Waldemar Hebisch for the reference [10].

3.2 Continuous-time Results

It is easy to check that the equality occurs if f is obtained by a translation, dilation or modulation of the Gaussian $g(x)=e^{-\frac{\pi}{2}x^2}$. Hirschman has conjectured that $H_{\frac{1}{2}}(f)$ is minimial (i.e. that $H_{\frac{1}{2}}(f)=\frac{n}{2}\ln\left(\frac{e}{2}\right)$ after the work of Beckner) only for these functions, [6]. As far as we know, this conjecture is still open.



a) b) c)

Figure 2. Non-optimality of a discretized Gaussian pulse: a) $N=4$ b) $N=16$ c) $N=256$

4. COMPARISONS

The most important relevant issue that we wish to make clear in this paper is that the discrete- and continuous-time versions of the Hirschman uncertainty principle yield solutions that are different, and significantly so. The discrete-time (unique) solution is distinctly non-Gaussian, as exemplified in Figure 1. In the discrete-time case, we have measured the non-optimality of the Gaussian solution in several instances. For a sampled zero-mean and unit-variance Gaussian pulse, we have the results for varying N shown in Figure 2. Note that the minimum of the main theorem is the solid line in each of the plots in Figure 2, and that, of course, our HOT basis function achieves this minimum for all valid p, i.e. $0 \le p \le 1$. As noted in [4], the value of N where the "cross-over" takes place depends on the variance of the discretized Gaussian pulse.

5. SUMMARY

In this paper, we have shown that the Hirschman uncertainty principle, when developed in the continuous- and discrete-time cases, leads to completely different optimal solutions. As we have seen, we have proven in [5] that the uniquely optimal solution for the discrete-time Hirschman uncertainty principle is a class of signals derived from [12]. These signals are not shaped as Gaussian pulses. In fact, as we have seen, discretized Gaussian pulses are not optimal. However, one optimal solution for the continuous-time case is a Gaussian pulse. Here, it is apparent that the premise commonly used in signal processing, namely that if we sample densely enough, the discrete-time case converges to the continuous-time case, is not correct in this instance. This lack of transferance across the "sampling" process could indicate a larger problem – We suggest that more work in this important area be performed to determine what impact this has, and to find out how widespread this problem might be.

6. REFERENCES

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