JOINT UNCERTAINTY MEASURES FOR MAXIMALLY DECIMATED *M*-CHANNEL PRIME FACTOR CASCADED WAVELET FILTER BANKS

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ABSTRACT

In this paper we study the joint time-frequency localization of cascade connections of maximally decimated wavelet filter banks. An M-channel bank is created by cascading a series of p_i-channel maximally decimated filter banks where $M = p_1 p_2 \cdots p_k$ is the prime factorization of M. Joint localization of the overall M-channel filter bank is quantified in terms of the geometric mean of certain discrete domain uncertainty measures of the analysis filters. As examples, we obtain interesting quantitative time-frequency localization measures for several two, four, and eight channel filter-banks.

1. INTRODUCTION

In [1], D. Gabor defined a measure of uncertainty for finite energy continuous signal as the product of the normalized signal's standard deviation in time and the standard deviation of the continuous Fourier transform of the normalized signal. This measure quantified the joint localization of a continuous signal in time and frequency. The lower bound of this was shown to be one-half when frequency is measured in units of Hertz. This relation is better known as the Heisenberg-Weyl Uncertainty Principle (HWUP) and the application of this principle in quantum mechanics is published in [2, 3]. For continuous, finite energy signals, the lower uncertainty bound is uniquely attained by the family of modulated and translated Gaussian functions, commonly referred to as the Gabor functions.

In the finite discrete domain, it has consequently been often assumed that sampled and truncated Gabor functions also possess good localization properties [4]. Such sampled, truncated Gabor functions have been used to design filter banks for applications in which good conjoint localization is desired [5]. Although filter banks of this type can be designed to have the perfect reconstruction property, the optimality of their joint localization in the class of finite length discrete signals is questionable. It is necessary to define a measure of uncertainty for finite length discrete filters analoguous to HWUP. In addition, the measure should be restricted to filters possessing a property which relates to continuous functions. The uncertainty measure and application to wavelets are described in [6]. As an extension of our work we propose a quantification of joint localization, *i.e.* uncertainty, for filter banks.

In this paper we will construct M-channel maximally decimated parallel filter banks (MDPFB's) by cascading prime channel MDPFB's. Since odd channel MDPFB's which admit perfect reconstruction are not known except in the trivial case of all-pass analysis and synthesis filters that are pure delays [7], we restrict our attention to M-channel filter banks where M is a power of two. We obtain several interesting results using well known two-channel orthogonal wavelet quadrature mirror filter banks and a biorthogonal wavelet filter bank in which the scaling function exhibits excellent conjoint localization.

2. FILTER BANKS

Many applications in coding and signal and image processing require an M-channel MDPFB. An excellent description of such an M-channel quadrature mirror MDPFB as shown in Fig. 1 can be in found in [7]. This filter bank requires M

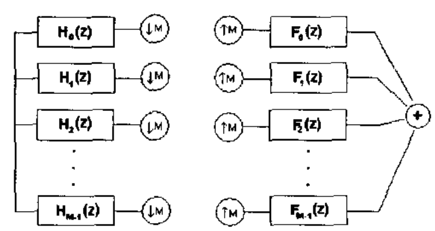

Fig. 1. M-channel maximally decimated filter bank.

analysis filters $H_0(z), H_1(z), \ldots, H_{M-1}(z)$ and M synthesis filters $F_0(z), F_1(z), \ldots, F_{M-1}(z)$. The output of each analysis filter is decimated by a factor of M, *i.e.*, output samples with time indices that are not integer multiples of M are discarded. Conversely, the input signals to the synthesis filters are interpolated (upsampled) by a factor of M: $M-1$ zeros are inserted between each sample prior to filtering. Provided the filter bank has the perfect reconstruction property, the sum of the outputs of the synthesis filters is a delayed version of the original filter bank input signal.

If $M = p_1 p_2 \cdots p_k$ is the prime factorization of M, then the filterbank in Fig 1 can be implemented by successively cascading p_i-channel MDFB's via the Noble identities [8]. Fig. 2 illustrates an analysis q-channel filter bank cascaded with an analysis p-channel filter bank. Similarly, Fig. 3 illustrates a synthesis p-channel filter bank cascaded with a synthesis q-channel filter bank. The Noble identities state that *a)* decimating a signal by a factor of M and subsequently convolving with filter $G(z)$ is equivalent to convolving with $G(z^M)$ and subsequently decimating the output by a factor of M and *b)* that convolving a signal with $G(z)$ and subsequently interpolating by a factor of M is equivalent to interpolating by factor

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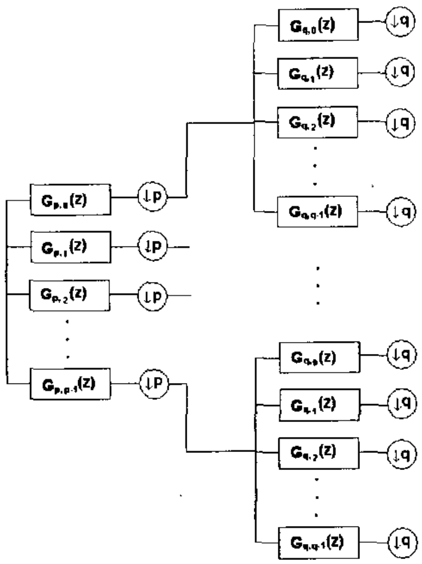

Fig. 2. An analysis q-channel maximally decimated filter bank successively cascaded to an analysis p-channel maximally decimated filter bank.

of M and subsequently convolving with $G(z^M)$. These identities are illustrated pictorially in Fig. 4.

Thus the cascaded analysis filter bank can be expressed as an M-channel parallel analysis filter bank as shown in Fig. 1 with

$$\begin{aligned} H_0(z) &= G_{p_1,0}(z)\cdots G_{p_k,0}(z^{p_1\cdots p_{k-1}}) \\ H_1(z) &= G_{p_1,0}(z)\cdots G_{p_k,1}(z^{p_1\cdots p_{k-1}}) \\ H_2(z) &= G_{p_1,0}(z)\cdots G_{p_k,2}(z^{p_1\cdots p_{k-1}}) \\ &\vdots \\ H_{M-1}(z) &= G_{p_1,p-1}(z)\cdots G_{p_k,p_k-1}(z^{p_1\cdots p_{k-1}}). \end{aligned}$$

The cascaded synthesis filter bank can be expressed as a M-channel parallel synthesis filter bank as described by:

$$\begin{aligned} F_0(z) &= \widetilde{G}_{p_k,0}(z^{p_1\cdots p_{k-1}})\cdots \widetilde{G}_{p_1,0}(z) \\ F_1(z) &= \widetilde{G}_{p_k,1}(z^{p_1\cdots p_{k-1}})\cdots \widetilde{G}_{p_1,0}(z) \\ F_2(z) &= \widetilde{G}_{p_k,2}(z^{p_1\cdots p_{k-1}})\cdots \widetilde{G}_{p_1,0}(z) \\ &\vdots \\ F_{M-1}(z) &= \widetilde{G}_{p_k,p_k-1}(z^{p_1\cdots p_{k-1}})\cdots \widetilde{G}_{p_1,0}(z). \end{aligned}$$

When M is a power of two, the M-channel MDPFB can be implemented by successively cascading two-channel wavelet filter banks. A wavelet filter bank can be considered as a two-channel MDFB in which there are two analysis filters and two synthesis filters. The two analysis (and the two synthesis) filters are half-band filters which are quadrature mirrors of each other in the orthogonal case.

It is well-known that FIR digital filters which are admissible as orthogonal or biorthogonal wavelets correspond to a continuous

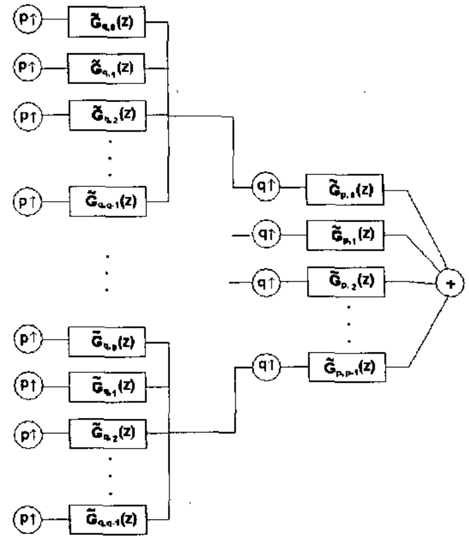

Fig. 3. A synthesis q-channel maximally decimated filter bank successively cascaded to a synthesis p-channel maximally decimated filter bank.

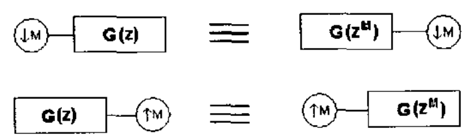

Fig. 4. The Noble identities.

finitely supported function where the set of dilations and translations constitute a basis for $L^2(\mathbb{R})$ [9, 10]. In addition, infinite-fold convolution of "*regular*" FIR digital wavelets with themselves converges to a compactly supported continuous function for which a suitable set of dilations and translations also form a basis for $L^2(\mathbb{R})$ [11, 12].

3. DEFINING DISCRETE UNCERTAINTY

In our previous paper [6] we describe an uncertainty measure for finite impulse response (FIR) digital filters. This measure quantifies the filter's localization in both finite discrete time and finite discrete frequency domains and is invariant under translations and modulations. The measure strongly resembles the well-known Heisenberg-Weyl Uncertainty Principle for finite energy continuous signals as defined by Gabor in [1]. For a unit $\ell_2([0,N-1])$-norm, length N filter $g[n]$, the uncertainty is defined by

$$\gamma_{\mathrm{g}}^2 = \sigma_{n,[\mathrm{g}]}^2\sigma_{\omega,[\mathrm{g}]}^2 \tag{1}$$

where

$$\sigma_{n,[\mathrm{g}]}^2 = \min\left\{\sigma_{n,\mathrm{f}}^2 \;\middle|\; \mathrm{f}\in[\mathrm{g}]\right\}, \tag{2}$$

$$\sigma^2_{n,\mathbf{f}} = \sum_{n=0}^{N-1} (n-\mu)^2 |f[n]|^2, \quad (3)$$

$$\mu = \sum_{n=0}^{N-1} n|f[n]|^2, \quad (4)$$

$$\sigma^2_{\omega,[\mathbf{g}]} = \min\left\{\sigma^2_{\omega,\mathbf{f}} \;\middle|\; \mathbf{f}\in[\mathbf{g}]\right\}, \quad (5)$$

$$\sigma^2_{\omega,\mathbf{f}} = \frac{1}{N}\sum_{k=0}^{N-1}(n-\nu)^2|F[k]|^2, \quad (6)$$

$$F[k] = \sum_{n=0}^{N-1} f[n]e^{-j\frac{2\pi}{N}nk}, \quad (7)$$

$$\nu = \frac{1}{N}\sum_{k=0}^{N-1} n|F[k]|^2, \quad (8)$$

$$[\mathbf{g}] = \{f[n] \mid f[n]\sim g[n]\}, \quad (9)$$

and $f[n]\sim g[n]$ if and only if

$$f[n] = e^{j\frac{2\pi}{N}(qn+r)} f[(n-p) \bmod N] \quad (10)$$

for some $p,q,r\in\mathbb{Z}$. It is easy to show that the relation "$\sim$" is an equivalence relation and that the equivalence class $[\mathbf{g}]$ is well-defined. In addition the following theorem holds:

Theorem 1 $f[n]\sim g[n] \Longleftrightarrow F[k]\sim G[k]$.

The lower bound for the uncertainty in equation (1) when applied to the set of length-N orthogonal or biorthogonal wavelets where N is even was shown in [6] to be non-trivial for $N>2$, *i.e.*,

$$\gamma^2_{\mathbf{g}} > \begin{cases} \frac{1}{192}(N^2-4) & \text{for } N \text{ not divisible by 4} \\ \frac{1}{192}(N^2+8) & \text{for } N \text{ divisible by 4.} \end{cases} \quad (11)$$

The inequality in equation (11) is strict, and thus non-attainable.

We use the geometric mean of the uncertainties (1) of the analysis filters in an M-channel MDPFB to define an aggregate measure of joint localization for the overall MDPFB structure. Let $\widehat{h}_i[n] = \frac{1}{\|\mathbf{h}_i\|}h_i[n]$ be the $\ell_2([0,N-1])$ normalized real valued analysis filter of an M-channel MDPFB and denote the set of analysis filters by $\langle\mathbf{H}\rangle = \{\mathbf{h}_i \mid 0\le i\le M-1\}$. We quantify the localization of the overall M-channel MDPFB by

$$\Gamma_M(\langle\mathbf{H}\rangle) = \sqrt[M]{\gamma^2_{\widehat{\mathbf{h}}_0}\gamma^2_{\widehat{\mathbf{h}}_{M-1}}\prod_{i=1}^{M-2}\sigma^2_{n,[\widehat{\mathbf{h}}_i]}\widehat{\sigma}^2_{\omega,[\widehat{\mathbf{h}}_i]}} \quad (12)$$

where

$$\widehat{\sigma}^2_{\omega,[\widehat{\mathbf{h}}_i]} = \min_{l\in[0,\frac{N}{2}]}\left\{\frac{2}{N}\sum_{k=0}^{\frac{N}{2}}(k-\nu_l)^2|H_i[k-l]|^2\right\}$$

and $\nu_l = \frac{2}{N}\sum_{k=0}^{\frac{N}{2}} k|H_i[(k-l) \bmod \frac{N}{2}]|^2$.

4. RESULTS

This section presents some results for four and eight channel MDPFB's created by cascading a two channel MDPFB at two and three levels, *resp.* Let $G_{2,0}(z)$ and $G_{2,1}(z)$ be the z-transforms of the analysis filters of a two channel perfect reconstruction MDPFB and let $\widetilde{G}_{2,0}(z)$ and $\widetilde{G}_{2,1}(z)$ be the two synthesis filters. Then the eight analysis filters of an eight channel MDPFB are

$$\begin{aligned}
H_0(z) &= G_{2,0}(z)G_{2,0}(z^2)G_{2,0}(z^4)\\
H_1(z) &= G_{2,0}(z)G_{2,0}(z^2)G_{2,1}(z^4)\\
H_2(z) &= G_{2,0}(z)G_{2,1}(z^2)G_{2,0}(z^4)\\
H_3(z) &= G_{2,0}(z)G_{2,1}(z^2)G_{2,1}(z^4)\\
H_4(z) &= G_{2,1}(z)G_{2,0}(z^2)G_{2,0}(z^4)\\
H_5(z) &= G_{2,1}(z)G_{2,0}(z^2)G_{2,1}(z^4)\\
H_6(z) &= G_{2,1}(z)G_{2,1}(z^2)G_{2,0}(z^4)\\
H_7(z) &= G_{2,1}(z)G_{2,1}(z^2)G_{2,1}(z^4).
\end{aligned}$$

The eight synthesis filters are

$$\begin{aligned}
F_0(z) &= \widetilde{G}_{2,0}(z^4)\widetilde{G}_{2,0}(z^2)\widetilde{G}_{2,0}(z)\\
F_1(z) &= \widetilde{G}_{2,1}(z^4)\widetilde{G}_{2,0}(z^2)\widetilde{G}_{2,0}(z)\\
F_2(z) &= \widetilde{G}_{2,0}(z^4)\widetilde{G}_{2,1}(z^2)\widetilde{G}_{2,0}(z)\\
F_3(z) &= \widetilde{G}_{2,1}(z^4)\widetilde{G}_{2,1}(z^2)\widetilde{G}_{2,0}(z)\\
F_4(z) &= \widetilde{G}_{2,0}(z^4)\widetilde{G}_{2,0}(z^2)\widetilde{G}_{2,1}(z)\\
F_5(z) &= \widetilde{G}_{2,1}(z^4)\widetilde{G}_{2,0}(z^2)\widetilde{G}_{2,1}(z)\\
F_6(z) &= \widetilde{G}_{2,0}(z^4)\widetilde{G}_{2,1}(z^2)\widetilde{G}_{2,1}(z)\\
F_7(z) &= \widetilde{G}_{2,1}(z^4)\widetilde{G}_{2,1}(z^2)\widetilde{G}_{2,1}(z).
\end{aligned}$$

Table 1 lists the time samples of the length 8 scaling functions corresponding to two well-known orthogonal wavelet quadrature mirror filter banks. The maximally flat scaling function of Daubechies will be referred to as $D_{2,0}(z)$. The Daubechies least asymmetric scaling function will be denoted by $S_{2,0}(z)$. It is well-known [13] that the other filters which make up a perfect reconstruction two channel quadrature mirror filter bank satisfy

$$\begin{aligned}
D_{2,1}(z) &= D_{2,0}(-z^{-1})\\
\widetilde{D}_{2,0}(z) &= D_{2,0}(z^{-1})\\
\widetilde{D}_{2,1}(z) &= D_{2,0}(-z)
\end{aligned}$$

and likewise for $S_{2,1}(z)$, $\widetilde{S}_{2,0}(z)$, and $\widetilde{S}_{2,1}(z)$.

Let $T_{2,0}(z)$ be the scaling function corresponding to the biorthogonal, linear phase FIR digital analysis filter that was shown in [6] to minimize the uncertainty measure (1). A technique by M. Vetterli and D. Le Gall, which can be found in [11], provided a method to determine $T_{2,1}(z)$, $\widetilde{T}_{2,0}(z)$, and $\widetilde{T}_{2,1}(z)$ so that perfect reconstruction is possible. This requires solving

$$\begin{aligned}
\frac{1}{t_{2,0}[0]}T_{2,0}(z) &= 1+\alpha_3 z^{-1}+(\alpha_2\alpha_3+\alpha_1\alpha_2+\alpha_1)z^{-2}\\
&+ (\alpha_1\alpha_3+\alpha_1\alpha_2\alpha_3+\alpha_2)z^{-3}\\
&+ (\alpha_1\alpha_3+\alpha_1\alpha_2\alpha_3+\alpha_2)z^{-4}\\
&+ (\alpha_2\alpha_3+\alpha_1\alpha_2+\alpha_1)z^{-5}+\alpha_3 z^{-6}+z^{-7}
\end{aligned}$$

n	$D_{2,0}(z)$	$S_{2,0}(z)$	$T_{2,0}(z)$	$\widehat{T}_{2,1}(z)$
0	-0.0106	-0.0758	-0.0132	-0.0161
1	0.0329	-0.0296	-0.0284	-0.0348
2	0.0308	0.4976	0.0436	-0.0189
3	-0.1870	0.8037	0.7051	0.7058
4	-0.0280	0.2979	0.7051	-0.7058
5	0.6309	-0.0992	0.0436	0.0189
6	0.7148	-0.0126	-0.0284	0.0348
7	0.2304	0.0322	-0.0132	0.0161

Table 1. Length 8 analysis filters. $\widehat{T}_{2,1}(z)$ is $T_{2,1}(z)$ normalized.

M	$\Gamma_M(\langle\mathbf{D}\rangle)$	$\Gamma_M(\langle\mathbf{S}\rangle)$	$\Gamma_M(\langle\mathbf{T}\rangle)$
2	0.9117	0.6529	0.5050
4	9.3774	6.9180	7.2165
8	58.9643	43.7670	73.2164

Table 2. Uncertainty measures $\Gamma_M(\langle\mathbf{H}\rangle)$ for $M = 2,4$ and 8.

for α_1, α_2, and α_3. $T_{2,1}(z)$ may then be determined by

$$\begin{aligned} T_{2,1}(z) &= 1+\alpha_3 z^{-1}+(\alpha_2\alpha_3+\alpha_1\alpha_2-\alpha_1)z^{-2} \\ &+ (-\alpha_1\alpha_3+\alpha_1\alpha_2\alpha_3+\alpha_2)z^{-3} \\ &+ (\alpha_1\alpha_3-\alpha_1\alpha_2\alpha_3-\alpha_2)z^{-4} \\ &+ (-\alpha_2\alpha_3-\alpha_1\alpha_2+\alpha_1)z^{-5}-\alpha_3 z^{-6}-z^{-7} \end{aligned}$$

Letting $\widetilde{T}_{2,0}(z) = \frac{1}{s}T_{2,1}(-z^{-1})$ and $\widetilde{T}_{2,1}(z) = \frac{1}{s}T_{2,0}(-z^{-1})$, where $s = (1-\alpha_1^2)(1-\alpha_2^2)(1-\alpha_3^2)$, we arrive at a two channel MDPFB with the perfect reconstruction property.

When $M = 2$, let $\langle\mathbf{D}\rangle = \{D_{2,m}(z) \mid m = 1,2\}$, and likewise for $\langle\mathbf{S}\rangle$ and $\langle\mathbf{T}\rangle$. When $M = 4$, let $\langle\mathbf{D}\rangle = \{D_{2,m}(z)D_{2,n}(z^2) \mid m,n = 1,2\}$, and likewise for $\langle\mathbf{S}\rangle$ and $\langle\mathbf{T}\rangle$. When $M = 8$, let $\langle\mathbf{D}\rangle = \{D_{2,m}(z)D_{2,n}(z^2)D_{2,k}(z^4) \mid m,n,k = 1,2\}$, and likewise for $\langle\mathbf{S}\rangle$ and $\langle\mathbf{T}\rangle$. Table 2 lists the uncertainty measure $\Gamma_M(\langle\mathbf{H}\rangle)$ for $\langle\mathbf{H}\rangle = \langle\mathbf{D}\rangle,\langle\mathbf{S}\rangle,\langle\mathbf{T}\rangle$ and for $M = 2,4,8$. When $M = 2$, we have $\Gamma_2(\langle\mathbf{T}\rangle) < \Gamma_2(\langle\mathbf{S}\rangle) < \Gamma_2(\langle\mathbf{D}\rangle)$. With $M = 4$, we obtain $\Gamma_8(\langle\mathbf{S}\rangle) < \Gamma_8(\langle\mathbf{T}\rangle) < \Gamma_8(\langle\mathbf{D}\rangle)$. With $M = 8$, we obtain the result $\Gamma_8(\langle\mathbf{S}\rangle) < \Gamma_8(\langle\mathbf{D}\rangle) < \Gamma_8(\langle\mathbf{T}\rangle)$.

5. CONCLUSION

In this paper we defined an uncertainty measure for M-channel MDPFB's. This measure quantifies the joint localization of a filter bank implemented as a cascade connection of FIR filters. The proposed measure is based on a previous formulation of uncertainty for FIR digital filters that is analogous to the well-known uncertainty measure of D. Gabor. Our proposed measure is the geometric mean of the uncertainties of the analysis filters.

Since the two channel MDPFB has recently been well studied and since nontrivial odd channel filter banks which have the perfect reconstruction property are not known, we restricted our attention to constructing filter banks where the number of channels is a positive integer power of two. In particular, the M-channel filter bank where $M = 2^n$ for some positive integer n was implemented by cascading a two channel filter bank at n levels.

In the results section, we reported the uncertainty of several two, four, and eight channel MDPFB's. The filter banks were constructed using two well-know Daubechies' wavelet quadrature mirror filter banks. They were compared with a lesser known two channel biorthogonal perfect reconstruction filter bank in which the scaling function minimizes the joint uncertainty measure (1). The results are indeed unexpected. The two channel filter bank which attained smallest uncertainty for $M = 2$ and when cascaded at two levels to $M = 4$ channels or at three levels to $M = 8$ channels does not exhibit the minimum uncertainty.

6. REFERENCES

[1] D. Gabor, "Theory of communication," *J. Inst. Elect. Eng. London*, vol. 93, no. III, pp. 429–457, 1946.

[2] W. Heisenberg, "The physical content of quantum kinematics and mechanics," in *Quantum Theory and Measurement*, J. A. Wheeler and W. H. Wheeler, Eds., pp. 62–86. Princeton University Press, Princeton, NJ, 1983.

[3] Hermann Weyl, *The Theory of Groups and Quantum Mechanics*, Dover Publications, Inc., New York, NY, 1950.

[4] V. DeBrunner, M. Özaydin, T. Przebinda, and J. Havlicek, "The optimal solutions to the continuous- and discrete-time versions of the Hirschman uncertainty principle," in *Proc. IEEE Int'l. Conf. Acoust., Speech, Signal Proc.*, Istanbul, Turkey, Jun. 5-9 2000, pp. 81–84.

[5] J. P. Havlicek, A. C. Bovik, and D. Chen, "AM-FM image modeling and Gabor analysis," in *Visual Information Representation, Communication, and Image Processing*, C. W. Chen and Y. Zhang, Eds., pp. 343–385. Marcel Dekker, New York, 1999.

[6] P.C. Tay, J. P. Havlicek, and V. DeBrunner, "A novel translation and modulation invariant discrete-discrete uncertainty measure," in *Proc. IEEE Int'l. Conf. Acoust., Speech, Signal Proc.*, Orlando, FL, May 13-17 2002, pp. II 1461–1464.

[7] P.P. Vaidyanathan, "Theory and design of M-channel maximally decimated quadrature mirror filters with arbitrary M, having the perfect-reconstruction property," *IEEE Trans. Acoustics, Speech, and Signal Proc.*, vol. ASSP-35, no. 4, pp. 476–492, April 1987.

[8] P.P. Vaidyanathan, "Multirate digital filters, filter banks, polyphase networks, and application: a tutorial," *Proc. of the IEEE*, vol. 78, no. 1, pp. 56–93, Jan. 1990.

[9] I. Daubechies, "Orthonormal bases of compactly supported wavelets," *Commun. Pure Appl. Math.*, vol. 51, pp. 909–996, 1988.

[10] A. Cohen, I. Daubechies, and J.-C. Feauveau, "Biorthogonal bases of compactly supported wavelets," *Commun. Pure Appl. Math.*, vol. 55, pp. 485–560, 1992.

[11] M. Vetterli and D. Le Gall, "Perfect reconstruction FIR filter banks: some properties and factorizations," *IEEE Trans. Acoust., Speech, and Signal Proc.*, vol. 37, no. 7, pp. 1057–1071, July 1989.

[12] M. Vetterli and C. Herley, "Wavelets and filter banks: theory and design," *IEEE Trans. Signal Proc.*, vol. 40, no. 9, pp. 2207–2232, Sept. 1992.

[13] I. Daubechies, *Ten lectures on wavelets*, SIAM, Philadelphia, 1992.