A NOVEL TRANSLATION AND MODULATION INVARIANT DISCRETE-DISCRETE UNCERTAINTY MEASURE

P. Tay, J.P. Havlicek, and V. DeBrunner

School of Electrical and Computer Engineering
University of Oklahoma, Norman, OK USA
{ptay,joebob,vdebrunn}@ou.edu

ABSTRACT

The quantification of signal localization simultaneously in time and in frequency is fundamental to a variety of signal processing applications where time-frequency analysis is to be performed on nonstationary signals. In this paper, we develop novel joint localization measures defined on equivalence classes of finitely supported discrete-time signals. These measures bear strong analogies to the well-known continuous-time Heisenberg-Weyl inequality. In particular, they are invariant to signal translations and modulations and admit an intuitive interpretation in terms of the temporal and spectral variance of the signal energy. The new measures are used to design optimal wavelet quadrature mirror filter banks that exhibit improved localization relative to the Haar and Daubechies analysis filters.

1. INTRODUCTION

Quantifying the localization of a signal simultaneously in time and in frequency is of great interest in a growing array of nonstationary signal processing applications. For example, jointly localized signals can be used to perform time frequency analysis upon or to efficiently represent a nonstationary signal. Joint time-frequency localization has traditionally been characterized using the well-known Heisenberg-Weyl inequality. Let $x : \mathbb{R} \longrightarrow \mathbb{C}$ be continuous with $\lim_{t\to\pm\infty} x(t) = 0$ and denote the Fourier transform by $X(f) = \int_{\mathbb{R}} x(t)e^{-j2\pi ft}\,dt$. With $\langle t\rangle = \int_{\mathbb{R}} t|x(t)|^2\,dt/||x||^2_{L^2}$ the mean time and $\langle f\rangle = \int_{\mathbb{R}} f|X(f)|^2\,df/||X||^2_{L^2}$ the mean frequency, the Heisenberg-Weyl uncertainty relation (HUP) states that

$$\Delta_t\Delta_f \geq \frac{1}{2}, \tag{1}$$

where

$$\Delta_t = \frac{1}{||x||^2_{L^2}}\int_{\mathbb{R}}(t-\langle t\rangle)^2|x(t)|^2\,dt \tag{2}$$

quantifies the localization or duration of $x(t)$ in time and

$$\Delta_f = \frac{1}{||X||^2_{L^2}}\int_{\mathbb{R}}(f-\langle f\rangle)^2|X(f)|^2\,df \tag{3}$$

quantifies the localization or bandwidth of $X(f)$ in frequency. Equality in (1) is achieved uniquely by the Gabor elementary functions $\psi(t) = e^{-\alpha^2(t-t_0)^2}e^{j2\pi f_0 t+\phi}$, where $\alpha, t_0, f_0, \phi \in \mathbb{R}$ [1]. Due to their good joint localization properties, these functions have been widely used in signal and image processing applications requiring time-frequency analysis [2]. Moreover, their popularity extends even into the discrete domain where sampled Gaussians have been widely applied.

In this paper, we study the problem of quantifying time and frequency localization for finitely supported discrete-time signals with respect to the discrete Fourier transform (DFT). Since these signals are discrete in both time and frequency, we refer to the associated measures as *discrete-discrete uncertainty measures.* Whereas the uncertainty measures (2) and (3) are invariant under both translation $[x(t) \mapsto x(t-t_0)]$ and the dual operation of modulation $[x(t) \mapsto e^{j2\pi f_0 t}x(t)]$, their obvious discrete analogues are not. This has the unfortunate consequence that shifting a finitely supported discrete signal in time or in frequency generally changes its localization in both domains. We introduce a new discrete-discrete measure that bears desirable analogies to the HUP (1), but also achieves both translation and modulation invariance. The new measure is used to design an optimally localized wavelet quadrature mirror filterbank (QMF) in Section 4.

2. DISCRETE-DISCRETE MEASURES

The measures (1)-(3) are intuitively appealing because they admit an interpretation of localization in terms of variance. Specifically, we consider $|x(t)|^2/||x||^2_{L^2}$ and $|X(f)|^2/||X||^2_{L^2}$ respectively as probability density functions (pdf's) characterizing the distribution of signal energy in time and in frequency. The measures Δ_t and Δ_f then admit formal interpretations as statistical variance with respect to these pdf's. In this section, we review several discrete-discrete uncertainty measures that have been proposed previously in the literature. Let $x[n]$ be a finite length sequence mapping $[0, N-1]$ to $\mathbb{C}$ with DFT $X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}nk}$, $0 \leq k < N$. Time translation of $x[n]$ is defined by the circular shift operation $y[n] = x[(n-n_0)_N]$, where $(n-n_0)_N$ is given by $(n-n_0) \bmod N \in [0, N-1]$. Similarly, modulation of $x[n]$ is defined by $y[n] = e^{j\frac{2\pi}{N}k_0 n}x[n] \leftrightarrow X[(k-k_0)_N]$.

2.1. The Measure of Donoho and Stark

The following theorem gives a novel discrete-discrete uncertainty measure proposed by Donoho and Stark [3] that quantifies the joint localization of $x[n]$ in terms of counting measure on the support of $x[n]$ and of $X[k]$.

Theorem 1 *Let N_t be the cardinal number of the set $\{x[n] \mid x[n] \neq 0\}$ and N_ω be the cardinal number of the set $\{X[k] \mid X[k] \neq 0\}$. Then*

$$N_t N_\omega \geq N \tag{4}$$

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and

$$N_t + N_\omega \geq 2\sqrt{N}. \quad (5)$$

The Kronecker delta attains equality in (4) but not in (5). If $N = p^2$ is a perfect square, then the picket fence sequence

$$III_p^p[n] = \begin{cases} 1, & n = ip,\ 0 \leq i < p, \\ 0, & \text{otherwise}, \end{cases} \quad (6)$$

attains the lower bounds in both (4) and (5).

Intuitive interpretation of (4) and (5) is straightforward: the more points at which $x[n]$ ($X[k]$) is nonzero, the less localized is the sequence in time (frequency). These measures are invariant under both translation and modulation. Compared to the continuous domain measures (1)-(3), what has been lost in (4) and (5) is the analogy with statistical variance. While these measures lead to an elegant theory, N_t quantifies time localization without regard for the specific values of $x[n]$ or their distribution in time. With $N = 8$, for example, the sequences $x_1[n] = \delta[n] + \delta[n-1] + \delta[n-2]$, $x_2[n] = \delta[n] + \delta[n-2] + \delta[n-3]$, and $x_3[n] = \delta[n] + 2\delta[n-2] + \delta[n-4]$ all have the same time localization.

2.2. Discrete Hirschman Uncertainty

DeBrunner, Özaydin, and Przebinda [4] defined discrete-discrete uncertainty measures in terms of the entropy of $x[n]$ and of $X[k]$, where it was assumed without loss of generality that $||x||_{\ell^2} = 1$. The time localization of $x[n]$ was quantified by

$$H(x) = -\sum_{n=0}^{N-1} |x[n]|^2 \ln |x[n]|^2, \quad (7)$$

while the frequency localization was quantified by

$$H(X) = -\sum_{k=0}^{N-1} \frac{|X[k]|^2}{N} \ln \frac{|X[k]|^2}{N}. \quad (8)$$

For $0 \leq \lambda \leq 1$, the joint λ-uncertainty of the sequence was then defined by

$$H_\lambda(x) = \lambda H(x) + (1-\lambda) H(X). \quad (9)$$

In the special case $\lambda = 1/2$, (9) is known as *Hirschman uncertainty* and the following theorem is obtained.

Theorem 2 *Let* $\lambda = \frac{1}{2}$. *Then*

$$H_{\frac{1}{2}}(x) \geq \frac{1}{2} \ln N. \quad (10)$$

For $N = p^2$, the sequences of any discrete cosine transform (DCT) basis, $\delta[n]$ or $\frac{1}{\sqrt{p}} III_p^p[n]$, minimize $H_{\frac{1}{2}}$ and attain equality in (10), where $III_p^p[n]$ was defined in (6). In fact for any $0 \leq \lambda \leq 1$, the λ-uncertainty H_λ of the picket fence sequence $III_p^p[n]$ is $\frac{1}{2} \ln N$. This leads to a rich emerging theory that casts doubt on the idea that sampled Gabor functions possess optimal time-frequency localization if the sampling is done "fast enough" [5].

The measures (7)-(10) are invariant under both translation and modulation. They may be interpreted with regards to entropy as follows: a perfectly flat sequence $x[n]$ has maximum entropy and minimum temporal localization, whereas the sequence $x[n] = \delta[n]$ has minimal entropy and maximum time localization. In view of the fact that the logarithm transforms multiplication into addition, there is an intuitive connection between (10) and (1). However, like (4) and (5), (7)-(10) fail to consider temporal relationships between the values $x[k]$; arbitrary permutations of the signal have no effect on the time localization.

2.3. Balanced Uncertainty

Monro, *et. al.*, computed both time dispersion Δt^2 and bandwidth $\Delta\omega^2$ for a length N sequence $x[n]$ according to [6, 7]

$$\Delta\omega^2 = \frac{\pi^2}{3} + \frac{4}{||x||_{\ell^2}^2} \sum_{n=0}^{N-2} \sum_{m=n+1}^{N-1} \frac{-1^{m-n}}{(m-n)^2} x[m]x[n], \quad (11)$$

$$\Delta t^2 = \sum_{n=0}^{N-1} (n-\tau)^2 x^2[n], \quad (12)$$

where $\tau = (\sum_{n=0}^{N-1} nx[n])/(\sum_{n=0}^{N-1} x[n])$. Let $x_a[n]$ be the low-pass analysis filter in an orthogonal wavelet QMF structure and let Δt_a^2 and $\Delta\omega_a^2$ be the time dispersion and bandwidth of $x_a[n]$. The joint uncertainty of the associated QMF is quantified by [6]

$$M(k) = \Delta\omega_a^2 + k^2 \Delta t_a^2, \quad (13)$$

where k is a parameter that balances the relative importance of time and frequency resolution. A similar measure

$$M(k_2, k_3, k_4) = \Delta\omega_a^2 + k_2 \Delta t_a^2 + k_3 \Delta\omega_s^2 + k_4 \Delta t_s^2 \quad (14)$$

was defined for biorthogonal wavelet QMF's, where subscripts a and s denote the analysis and synthesis filters [7]. Discrete wavelet transform image compression algorithms were designed by optimizing (13) and (14) and the reconstructed images were judged favorably in psychovisual experiments.

The measures (11) and (12) are difficult to interpret intuitively and are neither translation nor modulation invariant (in fact, for a low-pass analysis filter $x[n]$, (12) fails to converge under the important modulation $(-1)^n$ which defines the corresponding hi-pass filter). Inequalities analogous to (1), (4), (5), and (10) have not been published for the measures (13) and (14).

3. DEFINING UNCERTAINTY ON EQUIVALENCE CLASSES

We begin by summarizing the desirable properties of the HUP (1)-(3): A) it is invariant to translations and modulations: intuitively, simple shifts in time and frequency should not affect localization; B) it admits an intuitively satisfying interpretation in terms of variance in time and frequency; C) the inequality (1) provides a meaningful lower bound on joint localization that is uniquely realized by a nontrivial family of functions.

For a sequence $x[n]$ mapping $[0, N-1] \longrightarrow \mathbb{C}$, naïve discretization of (2) produces

$$\sigma_n^2 = \frac{1}{||x||_{\ell^2}^2} \sum_{n=0}^{N-1} (n-\mu)^2 |x[n]|^2, \quad (15)$$

where $\mu = (\sum_{n=0}^{N-1} n|x[n]|^2)/||x||_{\ell^2}^2$, while (3) yields

$$\sigma_\omega^2 = \frac{1}{||X||_{\ell^2}^2} \sum_{k=0}^{N-1} (k-\nu)^2 |X[k]|^2, \quad (16)$$

where $\nu = (\sum_{k=0}^{N-1} k|X[n]|^2)/||X||_{\ell^2}^2$. Although the localization measures (15) and (16) admit an interpretation in terms of statistical variance, they are not invariant under translations and modulations. This difficulty can be overcome by defining the measures not on sequences, but rather on *equivalence classes* of sequences.

Define a relation between two length N sequences $f[n]$ and $g[n]$ by $f \sim g$ if $\exists\, p, q, r \in \mathbb{Z}$ such that

$$g[n] = e^{j\frac{2\pi}{N}(qn+r)} f[(n-p)_N]. \tag{17}$$

It is trivial to show that the relation $\sim$ is reflexive, symmetric, and transitive and therefore defines an equivalence relation on the set of length N sequences. Therefore, for the sequence $f[n]$, we define the equivalence class $[f] = \{g[n] | g \sim f\}$.

Theorem 3 *Let $f[n]$ and $g[n]$ be two length N sequences. Then $f \sim g$ if and only if $F \sim G$.*

The preceding theorem, proof of which is omitted for brevity, suggests that, for a sequence $x[n]$, localization in time and frequency may be quantified by the measures

$$\left[\sigma_n^2\right] = \min_{[f]} \sigma_n^2, \tag{18}$$

$$\left[\sigma_\omega^2\right] = \min_{[F]} \sigma_\omega^2. \tag{19}$$

Note that the measures (18) and (19) are both translation and modulation invariant. However, they also satisfy the somewhat disappointing inequality $\left[\sigma_n^2\right]\left[\sigma_\omega^2\right] \geq 0$. A few of the sequences which achieve equality are the Kronecker delta, the constant sequence, and the sequence obtained by $\cos[\pi n]$. In the next section, we illustrate the useful application of related localization measures over a more restricted class of real-valued signals.

4. EXAMPLE

In this section we apply discrete-discrete uncertainty measures defined on equivalence classes to design an optimally localized low-pass analysis filter $f[n]$ for a wavelet QMF. Let $f : [0, N-1] \longrightarrow \mathbb{R}$. In addition, we require that N be even, $||f||_{\ell^2} = 1$, $\sum_{n=0}^{N-1} f[n] = \sqrt{2}$, and $F[N/2] = 0$, all of which are well-known conditions [8]. Note that these conditions exclude the Kronecker delta and the constant sequence from consideration. We have that

$$\sum_{n=0}^{N-1} f^2[n] = 1 = \frac{1}{N}\sum_{k=0}^{N-1} |F[k]|^2 \tag{20}$$

and also, since $|F[k]| = |F[-k \bmod N]|\ \forall\, k \in [0, N-1]$, that (16) may be simplified to

$$\sigma_\omega^2 = \frac{2}{N}\sum_{k=0}^{\frac{N}{2}} k^2 |F[k]|^2. \tag{21}$$

A numerical optimization was implemented to determine a low-pass scaling function $f[n]$ minimizing the product $\left[\sigma_n^2\right]\left[\sigma_\omega^2\right]$. For the cases $N = 2$ and $N = 4$, the Haar scaling function is the only admissible choice for $f[n]$. Note that this is the only known low-pass FIR analysis filter which possesses linear phase, exact reconstruction, and orthogonality. Optimizing the phase of $F[k]$ over a wide variety of fixed magnitude responses, we observed that the minimum uncertainty filter with respect to the measures (18) and (19) always had a linear phase given by

$$\varphi[k] = \begin{cases} \frac{\pi}{N}k & \text{for } k = 0, 1, \ldots, \frac{N}{2}, \\ -\frac{\pi}{N}k & \text{for } k = \frac{N}{2}+1, \ldots, N-1, \end{cases} \tag{22}$$

	Uncertainty			
Filter Length N	Optimal Lenght N Filter	Optimal Length $N-2$ Filter, Padded	Length N Haar Filter	Daubechies
2	0.00000	0.0000	0.0000	0.00000
4	0.12500	0.1250	0.1250	0.15180
6	0.27590	0.2917	0.2917	0.39580
8	0.48430	0.4921	0.5214	0.99230

Table 1. Uncertainty measure for even filter lengths $2 \leq N \leq 8$. The second column gives the uncertainty $\left[\sigma_n^2\right]\left[\sigma_\omega^2\right]$ for the optimal filter designed by the technique described in Section 4. The third column gives the uncertainty of the initial filter for each length N, which was obtained by zero padding the optimal filter found for length $N-2$. For comparison, the last two columns give uncertainty measures for the Haar and Daubechies low-pass analysis filters of corresponding length.

which coincides with the spectral phase of the Haar scaling function. Note that linear phase assures that the wavelet quadrature mirror filter bank can be cascaded to achieve different resolutions without the need for phase compensation.

For each $N > 4$, the search was initialized by zero padding the optimal solution from the length $N-2$ case. In the interest of computational tractability, the form (22) was assumed for the phase so that

$$F[k] = M[k]e^{-j\varphi[k]}, \tag{23}$$

which implies a symmetric impulse response $f[n]$. The numerical procedure then used a variational approach to determine the optimal magnitude response.

The optimal length $N = 6$ filter may be interpreted as a generalization of the Haar function that relaxes the conditions of perfect reconstruction and orthogonality to achieve improved joint localization. The low-pass analysis filter coefficients are given by

$$\begin{aligned} f[0] &= -0.0308556756313 = f[5] \\ f[1] &= 0.03226648753395 = f[4] \\ f[2] &= 0.70569596928391 = f[3] \end{aligned}$$

For this filter, the joint uncertainty is

$$\left[\sigma_n^2\right]\left[\sigma_\omega^2\right] = 0.27598499451912.$$

The experimental results are summarized in Table 1, which, for each even filter length N, gives the uncertainty of the optimal filter, the uncertainty of the zero padded optimal length $N-2$ filter (used to initialize the numerical optimization), and the uncertainties of the corresponding length Haar and Daubechies low-pass analysis filters. As can be seen from the table, the optimal filters designed by the procedure described in this section exhibit significantly better joint localization than the corresponding length Haar and Daubechies filters for lengths $4 \leq N \leq 8$.

5. CONCLUSION

The Heisenberg-Weyl inequality is appealing because it defines for continuous-time signals uncertainty measures that are translation and modulation invariant and also admit an intuitive interpretation in terms of statistical variance of the signal energy in time and in frequency. In this paper we developed novel analogous measures for finitely supported discrete-time signals. The key to obtaining these discrete-discrete measures was to define them on equivalence classes of signals rather than on the signals themselves. The new measures were used to design optimally localized wavelet quadrature mirror filter banks where the low-pass analysis filters exhibited significantly improved joint localization as compared to the low-pass Haar and Daubechies analysis filters. Our future research will focus on determining meaningful uncertainty bounds for these measures over restricted signal classes and on the development of improved numerical techniques for obtaining the optimal filters.

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