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FAST, EFFICIENT MEDIAN FILTERS WITH EVEN LENGTH WINDOWS

Indexing terms: Median filters, Programmable gate arrays

Median filtering has been extensively used in signal and image processing. Treatment has been limited to cases where the filter window spans an odd number of signal samples. The existence of cases where the median of an even number of samples is more efficient is demonstrated and an approach for designing real-time filters is presented.

Introduction: The robustness of the median against outliers means that median filtering is a particularly effective location estimator in the presence of signals and noise with heavily tailed densities. It has been used for a variety of applications in signal and image processing since its introduction in the early 1970s by Tukey.[1,2] The median filter with window W comprising N samples of the input sequence $\{x_i\}$ is a running nonlinear operator which produces output sample, y_i, equal to the median of the samples in W when the window is positioned at x_i. When N is odd, y_i takes the value of that sample in W such that $N - 1/2$ of the samples in W are greater than or equal to y_i and $N - 1/2$ of the samples in W are less than or equal to y_i. When N is even, y_i is defined as the arithmetic mean of the central two values among the samples in W. The required ordering of the samples in W is a highly nonlinear operation, and consequently the theoretical treatment and implementation of median filters are both difficult.

Additional complications arise when N is even. A symmetric contiguous window cannot be constructed. Computation of the even-point median requires that two order statistics be extracted from something? The latter fact significantly complicates analysis, and typically increases the algorithmic or hardware complexity of the filter by a factor of two. Although the median filter has been treated extensively in the literature, most authors have restricted their attention to the case of N odd. A comprehensive bibliography on the subject has been compiled by Coyle.[3]

Even-length windows: The efficiency of an estimator is inversely proportional to its variance. In general, smaller variance implies a more accurate estimator. For symmetrical densities, Hodges and Lehmann pointed out that the median of $N - 1$ is just as accurate as the median of N, where N is odd.[4] Suppose that $X_1, \ldots, X_N$ are N independent, identically distributed symmetric zero mean random variables with density $f(x)$ and distribution $F(x)$. If N is odd, the variance of the median $\tilde{X}$ is

$$\operatorname{var}(\tilde{X}) = \frac{N!}{\left[\frac{N}{2}\right]!\left[\frac{N}{2}\right]!} \int_{-\infty}^{\infty} x^2[F(x) - F^2(x)]^{[N/2]}f(x)\,dx \quad (1)$$

where $[x]$ is the largest integer not greater than x. When N is even, the joint density of the two central order statistics must be integrated. In this case, the variance of the median is given by

$$\operatorname{var}(\tilde{X}) = \frac{N!}{4m!\,m!} \int_{-\infty}^{\infty} \int_{-\infty}^{x_2} (x_1 + x_2)^2 F^m(x_1) \times [1 - F(x_2)]^m f(x_1) f(x_2)\,dx_1\,dx_2 \quad (2)$$

where $m = (N - 2) - 1$. For unit variance Gaussian variables, eqns. 1 and 2 must be numerically integrated. The results for several values of N are shown in Table 1 where it is observed that the variance decreases monotonically with increasing N. Such monotonically decreasing variance is also observed for Cauchy, double exponential, and triangular variables. For uniform $U(-\frac{1}{2}, \frac{1}{2})$ variables, eqns. 1 and 2 can be analytically integrated. The result for odd N is

Table 1 VARIANCE OF THE MEDIAN OF N I.I.D. $N(0, 1)$ VARIABLES

N	var ($\tilde{X}$)	N	var ($\tilde{X}$)	N	var ($\tilde{X}$)
1	1·0000	5	0·2868	9	0·1661
2	0·5000	6	0·2147	10	0·1383
3	0·4487	7	0·2104	11	0·1372
4	0·2982	8	0·1682	12	0·1175

$$\operatorname{var}(\tilde{X}) = \frac{1}{4(N + 2)} \quad (3)$$

for N even

$$\operatorname{var}(\tilde{X}) = \frac{N}{4(N + 1)(N + 2)} \quad (4)$$

Clearly, for uniform variables and N even, the median of N has lower variance than the medians of both $N - 1$ and $N + 1$. We are currently continuing to construct other distributions which display such non-monotone behaviour. Given that under certain circumstances one can improve the efficiency of a median filter estimator by choosing N even, the question naturally arises as to how such a filter can be made to run in real-time.

Fast realisation: The most difficult aspect of realising a fast median filter is the design of a mechanism for extracting the required order statistics. One structure capable of providing this mechanism is the selection network. Selection networks are arrays of compare and exchange (CEX) modules which partially sort their inputs using a fixed sequence of hardware comparisons. Although theoretical lower bounds on the number of CEXs required to find the median have been proven, exact optimal solutions are known only for specific small values of N. Fig. 1 shows a network which finds the two

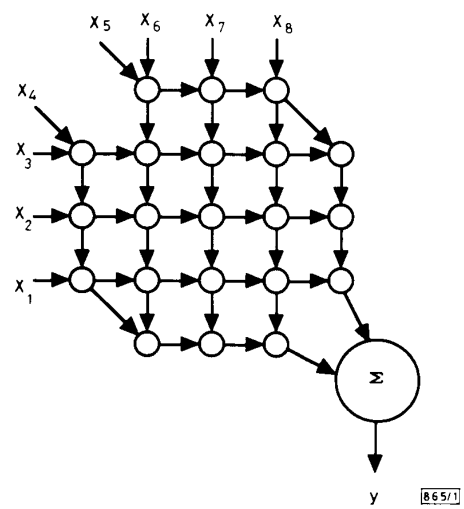

Fig. 1 *Selection network to find median of eight inputs*

central order statistics of eight inputs using 21 CEXs. The topology of this structure can be extended to find the median using $(N^2/4) + N - 3$ CEXs when N is even. A similar structure finds the median using $[(N + 1)^2/2] - 1$ CEXs when N is odd. Fig. 2 shows the space-time product, defined as the number of gates times the propagation delay (measured in gate delays), as a function of N and the word length (in bits) for selection networks using the topology of Fig. 1 and the hardware efficient bit-serial CEX units proposed by Knuth.[5] These units offer the advantage that hardware complexity is independent of word length. We have used this approach to realise the sorting network for an eight-point median filter using only 50% of a very modestly sized (3000 gate equivalent) field programmable gate array. The circuit accepts 12 bit inputs and extracts medians at 2·75 MHz.

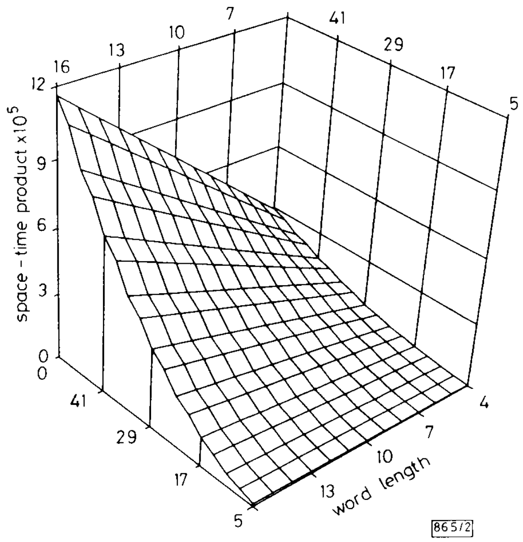

Fig. 2 *Space-time product for topologically extensible selection networks*

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FEMTOSECOND FIBRE LASER

Indexing terms: Optical fibres, Lasers, Pulse generation

Pulses shorter than 500 fs were generated by pulse compression modelocking of a neodymium fibre laser. An analysis of the pulse forming process is presented. Higher-order nonlinear effects were observed.

Single-mode rare-earth-doped optical fibres[1] are an ideal medium for the generation of ultra-short pulses, since they allow the exploitation of nonlinear optical effects for very efficient pulse-compression mechanisms.[2] By employing pulse compression mode-locking (PCM), i.e., active mode-locking with intra-cavity pulse compression, pulses of 2·4 ps width have been obtained in a Nd^{3+} fibre laser.[3] With further optimisation of the PCM, bandwidth limited femtosecond pulses may be produced in fibre lasers without the need for active stabilisation. We present a simple numerical analysis of the pulse forming process.

The set-up for PCM is shown in Fig. 1, where the amplifier is a short length of rare-earth-doped fibre; a spliced-on undoped dummy fibre provides positive group-velocity dispersion (PGVD) and self-phase modulation (SPM) and a grating pair gives negative GVD (NGVD). The modulator is a highly efficient Raman–Nath modulator[4] producing modulation indices[5] up to 200. An efficient modulator is required for PCM since, at least for inhomogeneously broadened media, a relatively short start pulse is required for the passive pulse shortening process to set in. The passive mechanism then takes command and the pulse width becomes independent of the modulation depth of the modulator. This explains the unique stability of the system. In the steady-state an initially unchirped pulse obtains a positive linear chirp in the fibre by the interplay of PGVD and SPM.[6] The linear chirp produced in the fibre is compensated by the grating pair and pulse compression[6] is obtained. The additional frequency components generated by SPM in the fibre are then finally cut-off by the finite bandwidth of the amplifier[5] and the original pulse-width is retained, so that the process can repeat itself. Thus chirp compensation and not dispersion compensation is relevant to the pulse-forming process. This relatively simple mechanism allows a straight-forward modelling of the steady-state. Standard mode-locking theory for homogeneously broadened media[5] was applied. Only the even terms up to second-order in the expansion of the operator for the modulator and the amplifier were used. The fibre was accounted for by the inclusion of an additional operator, where the pulse evolution was calculated using the nonlinear Schrödinger equation.[6]

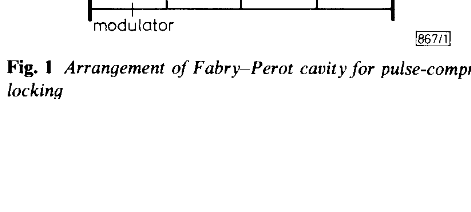

Fig. 1 *Arrangement of Fabry–Perot cavity for pulse-compression mode-locking*

The active fibre was silica based with a doping level of 1800 ppm Nd^{3+} and inhomogeneously broadened, emitting at 1·064 μm. The length was 45 cm and the length of the matched dummy fibre was varied between 1 and 3·5 m. The core radius was $\approx 2{\cdot}5\,\mu$m and the cut-off wavelength 1 μm giving a PVGD of $\approx 30\,ps^2/km$. The modulator was operated at twice or four times the cavity resonant frequency. The pulse repetition rate was halved under mode-locked operation because of dynamic gain saturation. With a 20% output coupler the lasing threshold was < 10 mW of launched CW power at 752 nm and the slope efficiency was 8%. The passive pulse compression was not self-starting and a high modulation index of about five was required for the pulses to compress. Stable pulses could be obtained for average output powers ranging from 500 μW to 8 mW with only a weak dependence of pulse width on output power and fibre length. For output powers larger than 2 mW, cross-phase-modulation-induced modulation instability generated sharp well-defined sidebands in the spectrum of the pulses.[3] The pulses had exponential pulse tails and were transform limited. The shortest pulses were obtained when the NGVD of the grating pair was about three times larger than the PGVD from the fibre. Short pulses could not be obtained in the vicinity of the point of zero overall dispersion. The dependence of the FWHM pulse-width on calculated overall dispersion for a fibre length of 4 m is shown in Fig. 2. Satisfactory agreement between theory and