Theorems for Discrete Filtered Modulated Signals*

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Abstract - Useful approximations to the responses of discrete linear systems and certain discrete nonlinear systems are developed for input complex AM-FM signals of the form $s(m) = a(m) \exp[j\phi(m)]$. These are used to derive limits on simple AM-FM demodulation mechanisms related to the Teager-Kaiser operator.

I. INTRODUCTION

We develop useful approximations to the responses of certain discrete linear systems and related discrete nonlinear systems to complex AM-FM signals of the form

$$s(m) = a(m)\, e^{j\phi(m)} \tag{1}$$

where $a: \mathbf{Z} \to \mathbf{R}$ are samples of a continuously differentiable amplitude modulation (AM) function $a(t)$, and $\phi: \mathbf{Z} \to \mathbf{R}$ are samples of a continuously twice-differentiable frequency modulation (FM) function $\phi(t)$. We also supply tight approximation bounds stated in terms of the smoothness of $a(m)$ and $\phi(m)$ as expressed by certain (Sobolev) norms, and also in terms of the duration of the involved linear system function(s).

AM-FM functions of the form (1) have recently been effectively used to model nonstationary, yet locally coherent structures in speech signals, images and other multidimensional signals. For example, they have been successfully used in the analysis of textured images when combined with Gabor wavelet image decompositions [1]-[3] and/or certain nonlinear energy operators [4], [5]. Models of the form (1) have also been extensively applied to the analysis of speech formation [6]-[8].

In these applications it is often of interest to pass the signal of interest through a linear system, such as a bandpass filter or a bank of filters [3], in order to extract local frequency structure that the model (1) captures.

In the next section, theorems are given that approximate the responses of arbitrary square-summable discrete linear systems to inputs of the form (1), and also to (nonlinear) products of such responses. Although the results are general, we will be most interested in systems having impulse responses of the form

$$h(m) = w(m)\, e^{j\omega_c m} \tag{2}$$

where $w(m)$ is a real-valued low-pass function.

These results culminate in Section III to establish theorems that give approximations on certain nonlinear AM-FM demodulation mechanisms related to the Teager-Kaiser operator [4]-[10].

II. APPROXIMATIONS TO LINEAR SYSTEM RESPONSE AND PRODUCTS OF RESPONSES

In the first main result, we approximate the response

$$s_*(m) = h(m) * s(m) = \sum_{p \in \mathbf{Z}} h(p)\, s(m-p) \tag{3}$$

of a square-summable discrete linear system $h: \mathbf{Z} \to \mathbf{C}$ to an input of the form (1). The approximation is given by

$$\begin{aligned} \hat{s}_*(m) &= s(m) \cdot H\left[e^{j\dot{\phi}(m)}\right] \\ &= a(m)\, e^{j\phi(m)} \cdot H\left[e^{j\dot{\phi}(m)}\right] \end{aligned} \tag{4}$$

with

$$\dot{\phi}(m) = \frac{d}{dm}\phi(m)$$

and where

$$H(e^{j\omega}) = \sum_{m \in \mathbf{Z}} h(m) e^{-j\omega m}$$

is also denoted $h(m) \leftrightarrow H(e^{j\omega})$. The approximation (4), when valid, has considerable potential for the analysis of discrete linear systems that decompose AM-FM signals of the form (1), since it takes the same general form as the response of the system to a monochromatic signal

$$s(m) = A e^{j\Omega m} \tag{5}$$

except that in (4), the argument of the system function $H(\cdot)$ is time-varying. Indeed, for a monochromatic signal (5), the approximation is exact. In all other cases, the error in (4) will be bounded in Theorem 1, which will require the following definitions:

$$a_{\max} = \sup_{m \in \mathbf{Z}} |a(m)|,$$

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$$\Delta_k(h) = \left[\sum_{m \in \mathbf{Z}} m^{2k} |h(m)|^2 \right]^{1/2} = \sqrt{\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{d^k}{d\omega^k} H(e^{j\omega}) \right|^2 d\omega}$$

and

$$\mathcal{D}(a; m) = |\dot{a}(m)| + \sqrt{2\gamma} \int_{-\infty}^{m} |\dot{a}(s)|\, ds$$

where

$$\gamma = \sum_{p \in \mathbf{N}} \left(\frac{1}{p}\right)^2 = 1.644934 \ldots$$

is Riemann's Zeta function. The functionals $\Delta_k(h)$ measure the temporal localization of the filters $h(m)$, or equivalently, of the low-pass equivalents $w(m)$ in (2), since $\Delta_k(w) = \Delta_k(h)$. The smoothness functionals $\mathcal{D}(a; m)$ are time-dependent Sobolev (derivative) norms; the approximation error bounds expressed by the Theorems generally contain terms imposing smoothness of this type on both $a(m)$ and on $\dot{\phi}(m)$.

Theorem 1: Let $e_*(m) = |s_*(m) - \hat{s}_*(m)|$. Then

$$e_*(m) \le \Delta_1(h)\, \mathcal{D}(a; m) + a_{\max} \Delta_2(h)\, \mathcal{D}(\dot{\phi}; m).$$

Thus, the approximation error is bounded by expressions of the temporal localization of the filter $h(m)$ and the smoothness (or lack thereof) of $a(m)$ and $\dot{\phi}(m)$. The bound becomes tight as $s(m)$ becomes monochromatic in the sense of small $\mathcal{D}(a; m)$ and $\mathcal{D}(\dot{\phi}; m)$.

Next, we develop an approximation to the products of linear system responses. This has application for the analysis of nonlinear systems that incorporate square-law devices and other product nonlinearities. Denote the product of response and conjugate response (where superscript ' * ' denotes complex conjugate)

$$s_{**} = [h_1 * s] \cdot [h_2 * s]^*$$

of square-summable discrete linear systems $h_1: \mathbf{Z} \to \mathbf{C}$, $h_2: \mathbf{Z} \to \mathbf{C}$ to an input (1). The approximation is

$$\hat{s}_{**}(m) = a^2(m) \cdot H_1[e^{j\dot{\phi}(m)}] H_2[e^{j\dot{\phi}(m)}], \quad (6)$$

where $h_k(m) \leftrightarrow H_k(e^{j\omega})$, $k = 1, 2$. While (6) is not unexpected in view of (4) and the bound in Theorem 1, it is not possible to develop a useful bound on the error from the result of Theorem 1. However, the following Lemma does supply such a bound. First define:

$$\bar{h} = \sum_{p \in \mathbf{Z}} |h(p)|.$$

Lemma 1: Let $e_{**}(m) = |s_{**}(m) - \hat{s}_{**}(m)|$. Then

$$e_{**}(m) \le a_{\max} \left[\bar{h}_1 \Delta_1(h_2) + \bar{h}_2 \Delta_1(h_1)\right] \mathcal{D}(a; m) + a_{\max}^2 \left[\bar{h}_1 \Delta_2(h_2) + \bar{h}_2 \Delta_2(h_1)\right] \mathcal{D}(\dot{\phi}; m).$$

III. APPLICATION TO A NONLINEAR AM-FM ENERGY OPERATOR

We now explore some interesting applications that are of general utility in the analysis of nonstationary signals. For example, it follows from Lemma 1 that for a filtered signal $s_*(m)$ given by (3)

$$|s_*(m)|^2 \approx a^2(m) \cdot \left|H[e^{j\dot{\phi}(m)}]\right|^2$$

for sufficiently smooth $a(m)$ and $\dot{\phi}(m)$ and localized $h(m)$. We now apply these results to obtain limits on the discrete nonlinear operator

$$\Phi\{s_*(m)\} = |s_*(m)|^2 - \mathrm{Re}\{s_*(m+1) s_*(m-1)\}.$$

Note that we are applying the operator $\Phi\{\cdot\}$ to *filtered* versions of $s(m)$; from the more general result, bounds on unfiltered approximations can be obtained.

A real-valued version of the operator $\Phi\{\cdot\}$, in fact identical to $\Phi\{\cdot\}$ if the input is real, was first developed by Teager [9] and subsequently investigated by Kaiser [10], [11], is effective for AM-FM demodulation [6]-[8]. The complex operator $\Phi\{\cdot\}$ was first introduced in a multidimensional form in [4]. In practical applications it is generally necessary that the energy operator Φ be preceded by a linear filter (usually bandpass) in order to counteract the effects of noise [12], [13]. We have:

$$\hat{\Phi}\{s_*(m)\} = 2a^2(m) \left[\sin \dot{\phi}(m)\right]^2 \left|H[e^{j\dot{\phi}(m)}]\right|^2$$

The error in this approximation is bounded by the following theorem, where $h_{\pm}(m) = h(m \pm 1)$.

Theorem 2: Let $e_{\Phi}(m) = \left|\Phi\{s_*(m)\} - \hat{\Phi}\{s_*(m)\}\right|$.

Then

$$e_{\Phi}(m) \le a_{\max} \bar{h} \left[\Delta_1(h_+) + 2\Delta_1(h) + \Delta_1(h_-)\right] \mathcal{D}(a; m) + a_{\max}^2 \bar{h} \left[\Delta_2(h_+) + 2\Delta_2(h) + \Delta_2(h_-)\right] \mathcal{D}(\dot{\phi}; m)$$

This approximation $\hat{\Phi}\{\cdot\}$ of $\Phi\{\cdot\}$ and accompanying error bound complement the results given in [8] for the (real-valued) Teager-Kaiser operator.

Example 1: Suppose $h(m) = \delta(m)$ (no filter). Then $|H(e^{j\omega})| = 1$ for $\omega \in [0, 2\pi]$ and so:

$$\Phi\{s(m)\} \approx \hat{\Phi}\{s(m)\} = 2a^2(m)\,[\sin \dot{\phi}(m)]^2 \qquad (7)$$

By Theorem 2, the approximation error is bounded by:

$$e_\Phi(m) \;\le 2\, a_{\max}[\mathcal{D}(a;\, m) + a_{\max}\, \mathcal{D}(\dot{\phi};\, m)].$$

Example 2: Next we approximate the energy $\Phi\{\cdot\}$ of a difference signal: take $h(m) = \delta(m+1)-\delta(m-1)$. Then $|H(e^{j\omega})| = 2|\sin \omega|$ and the approximation is:

$$\Phi\{s(m+1) - s(m-1)\} \approx \hat{\Phi}\{s(m+1) - s(m-1)\} = 8a^2(m)\,[\sin \dot{\phi}(m)]^4 \qquad (8)$$

From Theorem 2, the error in (8) is bounded above by:

$$24a_{\max}[\mathcal{D}(a;\, m) + 3a_{\max}\, \mathcal{D}(\dot{\phi};\, m)].$$

By combining (7) and (8), $a(m)$ and $\dot{\phi}(m)$ can be estimated via an *energy separation algorithm* [6]-[8]:

$$a^2(m) \approx \frac{2\Phi^2\{s(m)\}}{\Phi\{s(m+1) - s(m-1)\}} \qquad (9)$$

$$[\sin \dot{\phi}(m)]^2 \approx \frac{\Phi\{s(m+1) - s(m-1)\}}{4\Phi\{s(m)\}}. \qquad (10)$$

Results similar to (9), (10) can be obtained for filtered signals. Instead we explore an even simpler algorithm where filtering is also analyzed - an optimal class of filters is derived using a discrete uncertainty principle criterion recently developed by Doroslovacki, Fan, and Djuric [14].

IV. APPLICATION TO A DISCRETE MULTI-BAND AM-FM DEMODULATION SCHEME

Suppose that the signal (1) is passed through a set of bandpass filters $h_n(m)$; $n = 1, \ldots, N$, yielding outputs

$$s_n(m) = h_n(m) * s(m).$$

While the frequency tesselation will not be explored here, the filters $h_n(m)$ are assumed unit-energy and to sample the spectrum $[0, \pi]$ sufficiently densely that a large response is assured at each m; see, e.g., [1]-[3], [12], [13]. At each time m, the maximum-amplitude response

$$y(m) = \max_{1 \le n \le N} s_n(m) \qquad (11)$$

is found; let the maximizing channel be $h(m)$. Thus $h(m)$ captures a large percentage of local AM-FM signal energy. To estimate this information, first define

$$z(m) = \frac{1}{2}\,[y(m+K) + y(m-K)] \qquad (12)$$

which is equivalent to filtering with

$$g(m) = \frac{1}{2}\,[h(m+K) + h(m-K)]. \qquad (13)$$

A simple AM-FM demodulation algorithm is then:

$$\dot{\phi}(m) \approx \hat{\dot{\phi}}(m) = \frac{1}{K} \cdot \mathrm{Cos}^{-1}\left[\frac{z(m)}{y(m)}\right] \qquad (14)$$

and

$$a(m) \approx \hat{a}(m) = \left|\frac{y(m)}{H[e^{j\hat{\dot{\phi}}(m)}]}\right|. \qquad (15)$$

In (12)-(15) a 2-point *difference,* rather than *average* of filter outputs can also be used. However, calculating $\hat{\dot{\phi}}$ in (14) then requires finding an unambiguous interpretation for $\sin^{-1}$ on $[0, \pi]$. For $K \le 1$, (12) also has the advantage that it resists high-frequency (noise), which is a problem with operators of this type [12], [13].

Approximations (14), (15) are supported by the following. By Theorem 1, $z(m) \approx \hat{z}(m)$, where

$$\hat{z}(m) = s(m) \cos[K\dot{\phi}(m)]\, H[e^{j\dot{\phi}(m)}] \qquad (16)$$

with an error bounded above by

$$e_z(m) = |z(m) - \hat{z}(m)| \le \Delta_1(g)\, \mathcal{D}(a;\, m) + a_{\max}\, \Delta_2(g)\, \mathcal{D}(\dot{\phi};\, m).$$

Note that since $|Ae^{jq} + Be^{-jq}| \le \sqrt{2}\,|A + B|$, then

$$\Delta_k(g) \le \sqrt{2}\, \Delta_k\left\{\frac{1}{2}\,[w(m+K) + w(m-K)]\right\} = \sqrt{2}\, \partial_{k,K}(w)$$

where

$$\partial_{k,K}(w) = \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{d^k}{d\omega^k}\cos(K\omega)W(e^{j\omega})\right|^2 d\omega}\,.$$

and $w(m) \leftrightarrow W(e^{j\omega})$. Selecting $w(m)$ so that $\partial_{k,K}(w)$ is small is a localization criterion controlling the accuracy of (16). In fact, $\partial_{k,K}(w) = 0$ for the solutions of the difference equation

$$w(m+K) + w(m-K) = C \cdot \delta(m)$$

given by $W(e^{j\omega}) \propto \sec(K\omega)$. Imposing smoothness criteria on $\hat{a}(m)$ and/or $\hat{\phi}(m)$ results in more interesting solutions, a strategy supported by the assumption that the (14), (15) compute smooth modulation components. A similar approach was taken in [3], where the AM-FM estimates were made smooth (via a variational strategy) by forcing $\mathcal{D}(\hat{a};\infty)$ and $\mathcal{D}(\hat{\phi};\infty)$ small.

Since $\hat{a}(m)$ and $\hat{\phi}(m)$ are computed in a rather direct manner from $y(m)$, a simple approach is to minimize

$$\frac{1}{4}\sum_{m\in\mathbf{Z}} |h(m+K) - h(m-K)|^2 \leq \sqrt{2}\,\Delta_0\left\{\frac{1}{2}\left[w(m+K) - w(m-K)\right]\right\} = \sqrt{2}\,\varepsilon_K(w),$$

where

$$\varepsilon_K(w) = \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin^2(K\omega)\left|W(e^{j\omega})\right|^2 d\omega}\,.$$

Simultaneously forcing $\partial_{k,K}(w)$ and $\varepsilon_K(w)$ to be small [for unit-energy $w(m)$] are conflicting goals as expressed by a form of the Doroslovacki-Fan-Djuric uncertainty relation [14], which states that for $K = 1/2$,

$$\partial_{k,K}(w)\cdot\varepsilon_K(w) \geq \frac{1}{4} \qquad (17)$$

The filters that uniquely achieve the lower bound in (17) have the form ($p > -1/2$) [14]:

$$W(e^{j\omega}) = B(p)\left[2\cos(\omega/2)\right]^p. \qquad (18)$$

where

$$B(p) = \frac{\Gamma(p+1)}{\sqrt{\Gamma(2p+1)}} \qquad (19)$$

yields unit energy. The optimal filters (18) maintain localized low-frequency energy while simultaneously de-emphasizing high-frequency energy. These filters approach a Gaussian characteristic as $p \to \infty$ [14], hence the optimal channel filters

$$H(e^{j\omega}) = B(p)\left\{2\cos\left[(\omega-\omega_c)/2\right]\right\}^p. \qquad (20)$$

resemble Gabor functions [1]-[3], [5], [12]-[14] for large p, in agreement with the continuous formulation.

V. CONCLUDING REMARKS

AM-FM models such as (1) that capture physically meaningful signal nonstationarities are finding increased applications. New analysis techniques, expanding on those given here, will help to exploit the power of the approach. Currently, we are studying extended models of the form:

$$s(m) = \sum_k a_k(m)\, e^{j\phi_k(m)}. \qquad (19)$$

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