2D Phase Unwrapping

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0.1 What is Phase Unwrapping?

Phase unwrapping is an important process in many applications that take advantage of the phase information. In synthetic aperature radar (SAR) interferometric imaging, the phase value at a given point indicate the terrain evaluation height [1]. In fiber-optic interferometry, the phase value represent the depth of the imaged object. In magetic resonant (MR), the phase values contain information about flow or inhomogeneities in the magnetic field. However, in these applications, direct measurements of the phase functions are not possible. For example, in compenstated imaging, one can have only obtain phase differences from the receivers [2, 3]. In other applications, only the wrapped phase are measurable and they do not provide an intuitive way to present and perform analysis on the observed phenomena. Due to the emphasis of this dissertation on image processing, we will limit our discussion of the phase unwrapping to the 2-D signals.

The 2-D phase unwrapping process aims to find the unwrapped phase function from its wrapped function. In order words, given values of a continuous phase function by an inversion of a trigonometric function, for example, $\arccos(\phi(\mathbf{x}))$, we want to find $\phi(\mathbf{x})$ such that its range is not restricted in $[0, 2\pi]$. Formally, let $\phi(\mathbf{x})$ be a continuous phase function. Let $\mathcal{W}[.]$ be the wrap operator such that $\mathcal{W}[\phi(\mathbf{x})] \in [0, 2\pi]$. Given $\mathcal{W}[\phi(\mathbf{x})]$, we need to compute $\phi(\mathbf{x})$. Nevertheless, the 2-D phase unwrapping is an *ill-posed* problem. In practice, due to noise or missing samples at the sensor's receiver, a unique unwrapped phase solution can not be obtained. Therefore, one has to resort approximation algorithms.

INSERT FIGURES TO ILLUSTRATE THE WRAPPED PHASE vs. UNWRAPPED PHASE

There are four main phase unwrapping categories: path integration, energy norm minimization, model-based estimation, and bayesian based estimation. All approaches use either acquired or computed local phase differences, *i.e.*, gradient.

0.1.1 Numerical Path Integration

Goldstein, Zebker, and Werner [4] proposed a path integration phase unwrapping algorithm. They provided an algorithm, *branch cut* to detect local errors caused by large phase discontinutities and prevent them from contributing to the the global phase reconstruction. The unwrapped phase is then obtained by performing path integration by knowing local horizontal and vertical derivatives. The numerical integration process must not cross the cut boundaries detected by the *branch cuts* algorithm.

0.1.2 Least-square Energy Minimization

Fried [2] and Hudgin [3] formulated the least-square phase reconstruction problem for the wave-front sensor. They aimed to minimize the sum of errors between the phase differences and gradient of the unwrapped phase. Hunt [5] casted the phase reconstruction problem with linear algebra and proposed a method to improve the convergence rate of the phase solution. Takajo and Takahashi [6] justified the least-square phase reconstruction formulation and introduced conditions where the least-square phase solution is the unique solution. The authors then proposed a closed-form non-iterative algorithm in frequency domain to solve for the phase function [7]. Ghiglia and Romero [8] extended the least-square phase reconstruction to facilitate weighted constribution of measured phase difference. They proposed two iterative algorithms to solve for the phase. Strand, Taxt, and Jain proposed using the least-square phase unwrapping in small block [1]. Bioucas-Dias and Valadão [9] proposed an energy minimization framework for 2D phase unwrapping based on graph cuts. Spagnolini [10] used the IF estimation directly from the model instead of the wrapped phase. The estimated IF is then used in 2-D phase unwrapping using the least-square framework. Because the the unwrapped phase values differ from that of the wrapped by multiple of 2π, Costantini [11] formulated the phase unwrapping problem as the energy minimization with integer variables.

0.1.3 Model-based Parameter Estimation

Friedlander and Francos [12] used a parametric model for phase unwrapping. First, the authors used a 2-D polynomial model to fit the observed phase. The estimated phase is then used to direct the phase unwrapping process where the phase of each sample is corrected by adding/subtracting a multiple of 2π based on the difference between the principle value of the phase and the estimated phase. For arbitrary phase unwrapping, the observed wrapped phase signal is first segmented prior to the model fitting process.

0.1.4 Bayesian Phase Unwrapping

Nico, Palubinskas, and Datcu [13] applied the Bayesian framework to the phase unwrapping problem. Because measurement noise and phase aliasing caused the least-square phase reconstruction solution inaccurate, they advocated the use of first and second regularization term to enforce the phase pior model. However, it is not easy to find an suitable prior model any arbitrary phase function.

0.2 Least-square Phase Unwrapping for the AM-FM Image Model

In many applications such as SAR or fiber-optic interferometry, we can not measure the values of the phase function directly. Let $\mathbf{x} \in \mathbb{R}^2$. Let $\phi(\mathbf{x})$ be the true phase function that we want to estimate. Let $\nabla\rho(\mathbf{x})$ be the measured phase gradient. The least-square phase unwrapping approach finds the unwrapped phase $\phi(\mathbf{x})$ by minimizing the mean square error between the gradient of $\phi(\mathbf{x})$ and the measured phase differences $\nabla\rho(\mathbf{x})$. In other words, we can cast their relationship as

$$\nabla\phi(\mathbf{x}) = \nabla\rho(\mathbf{x}) + d(\mathbf{x}), \tag{1}$$

where $d(\mathbf{x})$ models the distortion during the measurement process.

In practice, acquired measurements are discrete. Therefore, the finite phase difference is often used to approximate the derivative operator. For example, $[\phi(m+1,n) - \phi(m,n)]$ and $[\phi(m,n+1) - \phi(m,n)]$ are approximation of vertical and horizontal derivative at location (m,n) in the grid $[0 \cdots M-1] \times [0 \cdots N-1]$. Specifically, let $\rho_m(m,n)$ and $\rho_n(m,n)$ be the measured phase gradient, *i.e.*, $\rho_m(m,n)$ and $\rho_n(m,n)$ be discrete approximation of the vertical and the horizontal derivative of the measured phase $\rho(m,n)$. Let $\nabla\phi(m,n) = [\phi_m(m,n)\ \phi_m(m,n)]^T$ be the gradient field of the unwrapped phase. The unwrapped phase

$\phi(m,n)$ is the solution to the L_2 norm minimization

$$\mathcal{E}(\phi(m,n)) = ||\phi_m(m,n) - \rho_m(m,n)||^2 + ||\phi_n(m,n) - \rho_n(m,n)||^2. \tag{2}$$

Hunt [5] reorganized (2) in term of matrix multiplication. He constructed the matrix $\mathbf{A}$ which acts like a phase difference operator. In this formulation, the phase functions $\phi(m,n)$, $\rho_m(m,n)$, and $\rho_m(m,n)$ are vectorized into 1-D vector. Concretely, we denote ϕ a 1-D vector of $\phi(m,n)$ and $\gamma = [\rho_m\ \rho_n]^T$ a 1-D vector consisting of two stacked 1-D vectors ρ_m, and ρ_n. The energy minimzation equation (2) is equivalent to

$$\begin{aligned}
\mathcal{E}(\phi) &= ||\mathbf{A}\phi - \gamma||^2 \\
&= (\mathbf{A}\phi - \gamma)^T(\mathbf{A}\phi - \gamma) \\
&= (\mathbf{A}\phi)^T\mathbf{A}\phi - (\mathbf{A}\phi)^T\gamma - \gamma^T\mathbf{A}\phi + \gamma^T\gamma \\
&= \phi^T\mathbf{A}^T\mathbf{A}\phi - 2\phi^T\mathbf{A}^T\gamma + \gamma^T\gamma.
\end{aligned} \tag{3}$$

We can see that (3) is a quadratice function of ϕ. The optimal solution for ϕ is computed by taking derivative of $\mathcal{E}(\phi)$ and set the derivative to zeros as

$$\begin{aligned}
\frac{\partial\mathcal{E}}{\partial\phi} = 2\mathbf{A}^T\mathbf{A}\phi - 2\mathbf{A}^T\gamma = \mathbf{0}. \\
\Rightarrow \mathbf{A}^T\mathbf{A}\phi = \mathbf{A}^T\gamma.
\end{aligned} \tag{4}$$

We notice that the solution in (4) is the well-known least-square solution of an overdetermined linear system. We can solve (4) for ϕ using matrix inversion provided that the matrix $\mathbf{A}^T\mathbf{A}$ is not singular. However, the matrix $\mathbf{A}$ in this problem is speciall matrix to compute discrete gradient. As a result, $\mathbf{A}^T\mathbf{A}\phi$ can be intepreted as the Laplacian of ϕ and $\mathbf{A}^T\rho$ represents the derivative of the measured gradient ρ, which is the Laplacian of measured phase ρ. With this representation, equation (4) is the discretization of the Poisson equation which can be solved exactly using the fast discrete cosine transform (DCT) [7]. Let Φ be the 2-D DCT transform of $\phi(m,n)$ and Γ be the 2-D DCT trasnform of $\gamma(m,n)$.

$$\begin{aligned}
\overbrace{\Big(\phi(m+1,n) - 2\phi(m,n) + \phi(m-1,n)\Big)}^{\phi_{mm}(m,n)} + \overbrace{\Big(\phi(m,n+1) - 2\phi(m,n) + \phi(m,n-1)\Big)}^{\phi_{nn}(m,n)} \quad = \\
\Big(\rho_m(m,n) - \rho_m(m-1,n)\Big) + \Big(\rho_n(m,n) - \rho_n(m,n-1)\Big)
\end{aligned} \tag{5}$$

$$\phi(m,n) = \sum_{i=0}^{M-1}\sum_{j=0}^{N-1} w(i,j)\Phi(i,j)\cos\left[\frac{\pi}{2M}i(2m+1)\right]\cos\left[\frac{\pi}{2N}j(2n+1)\right], \tag{6}$$

where

$$w(i,j) = \begin{cases} 0.25 & \text{if } i=j=0, \\ 0.5 & \text{if } i=0 \text{ and } j\neq 0, \\ 0.5 & \text{if } j=0 \text{ and } i\neq 0, \\ 1.0 & \text{otherwise.} \end{cases}$$

Substituting (6) to each element of the left hand side in (5) and expanding the first cosine term, we have

$$\begin{aligned}
\phi(m+1,n) &= \sum_{i=0}^{M-1}\sum_{j=0}^{N-1} w(i,j)\Phi(i,j)\cos\left[\frac{\pi}{2M}i(2(m+1)+1)\right]\cos\left[\frac{\pi}{2N}j(2n+1)\right] \\
&= \sum_{i=0}^{M-1}\sum_{j=0}^{N-1} w(i,j)\Phi(i,j)\cos\left[\frac{\pi}{2M}i(2m+1)+\frac{\pi}{M}i\right]\cos\left[\frac{\pi}{2N}j(2n+1)\right] \\
&= \sum_{i=0}^{M-1}\sum_{j=0}^{N-1} w(i,j)\Phi(i,j)\Bigg(\cos\left[\frac{\pi}{2M}i(2m+1)\right]\cos\left[\frac{\pi}{M}i\right]\cos\left[\frac{\pi}{2N}j(2n+1)\right] - \\
&\quad \sin\left[\frac{\pi}{2M}i(2m+1)\right]\sin\left[\frac{\pi}{M}i\right]\cos\left[\frac{\pi}{2N}j(2n+1)\right]\Bigg).
\end{aligned} \tag{7}$$

$$\begin{aligned}
\phi(m-1,n) &= \sum_{i=0}^{M-1}\sum_{j=0}^{N-1} w(i,j)\Phi(i,j)\cos\left[\frac{\pi}{2M}i(2(m-1)+1)\right]\cos\left[\frac{\pi}{2N}j(2n+1)\right] \\
&= \sum_{i=0}^{M-1}\sum_{j=0}^{N-1} w(i,j)\Phi(i,j)\cos\left[\frac{\pi}{2M}i(2m+1)-\frac{\pi}{M}i\right]\cos\left[\frac{\pi}{2N}j(2n+1)\right] \\
&= \sum_{i=0}^{M-1}\sum_{j=0}^{N-1} w(i,j)\Phi(i,j)\Bigg(\cos\left[\frac{\pi}{2M}i(2m+1)\right]\cos\left[\frac{\pi}{M}i\right]\cos\left[\frac{\pi}{2N}j(2n+1)\right] + \\
&\quad \sin\left[\frac{\pi}{2M}i(2m+1)\right]\sin\left[\frac{\pi}{M}i\right]\cos\left[\frac{\pi}{2N}j(2n+1)\right]\Bigg).
\end{aligned} \tag{8}$$

Repeat the same expansion for every element on the left hand side of (6) and simplify the overall sum, we arrive

$$\text{LHS} = \sum_{i=0}^{M-1}\sum_{j=0}^{N-1} w(i,j)\Phi(i,j)\cos\left[\frac{\pi}{2M}i(2m+1)\right]\cos\left[\frac{\pi}{2N}j(2n+1)\right]\left(2\cos\left[\frac{\pi}{M}i\right]+2\cos\left[\frac{\pi}{N}j\right]-4\right) \quad (9)$$

Repeat the same expansion for every element on the right hand side of (6) and simplify the overall sum, we arrive

$$\text{RHS} = \sum_{i=0}^{M-1}\sum_{j=0}^{N-1} w(i,j)\Gamma(i,j)\cos\left[\frac{\pi}{2M}i(2m+1)\right]\cos\left[\frac{\pi}{2N}j(2n+1)\right] \quad (10)$$

Compare (9) and (10), we have the relationship in DCT domain

$$\Phi(i,j) = \frac{\Gamma(i,j)}{2\cos\left(\frac{\pi}{M}i\right)+2\cos\left(\frac{\pi}{N}j\right)-4}. \quad (11)$$

The unwrapped phase solution is obtained by taking the inverse discrite cosine transform as

$$\phi(m,n) = \text{IDCT}\{\Phi\}. \quad (12)$$

$\Phi(0,0)$ in (11) is not defined because the denominator is 0. In practice, one can set $\Phi(0,0) = 0$ which results in a zero mean unwrapped phase function. Better yet, one can estimate the constant c such that it minimize the energy between the $\cos(\mathcal{W}\{\phi(m,n)\})$ and $\cos(\phi(m,n))$. In other words, one minimizes the following norm

$$\mathcal{E}(c) = ||\cos(\mathcal{W}\{\phi(m,n)\}) - \cos(\phi(m,n)+c)||^2. \quad (13)$$

Pritt [14] computed (13) with different c over the range of $[0\ 2\pi]$ and select choose c with the lowest error. One can also find the constant c using an interative approach such as gradient descent.

INSERT PICTURES HERE

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